

Exercise 1.1.4:

Let  $\mathcal{H} = \ell^2(\mathbb{Z})$  and  $f_n(j) = \begin{cases} 1, & \text{if } j=n \\ 0, & \text{otherwise.} \end{cases}$

Then for  $\forall g \in \mathcal{H}$ , let  $g = (g_1, g_2, \dots)$

$$\langle g, f_n \rangle = \langle g, f_n \rangle = \sum_{j \in \mathbb{Z}} g(j) f_n(j) = g(n) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

which implies that  $w - \lim_{n \rightarrow \infty} f_n = 0$

Then consider  $\|f_n - 0\| = \|f_n\| = 1 \neq 0$ .

which means that  $f_n$  is not strongly convergent to 0

Exercise 1.3.6.

Since  $\{A_n\}_{n \in \mathbb{N}} \subset \mathcal{B}(\mathcal{H})$  is a strongly convergent sequence and by the Uniform Boundedness Principle, we can know that  $\{A_n\}_{n \in \mathbb{N}}$  is bounded.

Assume there exists a constant  $M < \infty$ . s.t.  $\|A_n\| \leq M$  for all  $n \in \mathbb{N}$

Let  $f \in \mathcal{H}$  then

$$\begin{aligned} & \|A_\infty B_\infty f - A_n B_n f\| \\ &= \| (A_\infty - A_n) B_\infty f + A_n B_\infty f - A_n B_n f \| \\ &= \| (A_\infty - A_n) B_\infty f + A_n (B_\infty - B_n) f \| \end{aligned}$$

(triangle inequality)  $\leq \| (A_\infty - A_n) B_\infty f \| + \| A_n (B_\infty - B_n) f \|$

$$\leq \| (A_\infty - A_n) B_\infty f \| + M \| (B_\infty - B_n) f \| \quad (*)$$

Since  $s - \lim_{n \rightarrow \infty} B_n = B_\infty$ .  $s - \lim_{n \rightarrow \infty} A_n = A_\infty$ .

then  $\lim_{n \rightarrow \infty} \| (A_\infty - A_n) g \| = 0$ ,  $\lim_{n \rightarrow \infty} \| (B_\infty - B_n) f \| = 0$  for any  $f, g \in \mathcal{H}$ .

thus  $(*) = \| (A_\infty - A_n) B_\infty f \| + M \| (B_\infty - B_n) f \| \rightarrow 0$  as  $n \rightarrow \infty$ .

which implies that  $\{A_n B_n\}_{n \in \mathbb{N}}$  is strongly convergent to  $A_\infty B_\infty$ .

### Exercise 1.4.2.

$$\text{Set } T = \sup_{f \in M, \|f\|=1} |\langle f, Bf \rangle|$$

By Cauchy-Schwarz inequality we have

$$T = \sup_{f \in M, \|f\|=1} |\langle f, Bf \rangle| \leq \sup_{f \in M, \|f\|=1} \|f\| \cdot \|Bf\| \leq \sup_{f \in M, \|f\|=1} \|f\| \cdot \|B\| \cdot \|f\| = \|B\|$$

$$\text{so } T \leq \|B\| \quad \textcircled{1}$$

On the other hand,

Since for  $f, g \in M$  and  $\|f\| = \|g\| = 1$ .

$$\|B\| = \sup_{f, g \in M, \|f\| = \|g\| = 1} |\langle f, Bg \rangle|$$

so we need to show  $T \geq |\langle f, Bg \rangle|$  for all  $f, g \in M$  with  $\|f\| = \|g\| = 1$ .

$$\text{Since } |\langle f, Bg \rangle| = |\langle e^{i\alpha} f, Bg \rangle| \text{ for } \forall \alpha \in \mathbb{R}$$

and for fixed  $f, g$ , there  $\exists \alpha \in \mathbb{R}$ , s.t.  $\langle e^{i\alpha} f, Bg \rangle$  is real and non-negative

By the polarisation identity :

$$4\langle f, g \rangle = \|f+g\|^2 - \|f-g\|^2 - i\|f+ig\|^2 + i\|f-ig\|^2$$

we can get

$$4\langle f, Bg \rangle = \langle f+g, B(f+g) \rangle - \langle f-g, B(f-g) \rangle - i\langle f+ig, B(f+ig) \rangle + i\langle f-ig, B(f-ig) \rangle$$

Since  $B$  is self-adjoint, then for  $\forall f \in M$ ,  $\langle f, Bf \rangle$  is real,

so  $\text{Re}(4\langle f, Bg \rangle) = \langle f+g, B(f+g) \rangle - \langle f-g, B(f-g) \rangle$  is the real part.

$$\text{Since } \langle e^{i\alpha} f, Bg \rangle \in \mathbb{R}, \text{ and } |\langle f, Bg \rangle| = |\langle e^{i\alpha} f, Bg \rangle| \in \mathbb{R}$$

$$\text{thus } 4\langle f, Bg \rangle = \langle f+g, B(f+g) \rangle - \langle f-g, B(f-g) \rangle$$

$$\leq |\langle f+g, B(f+g) \rangle| + |\langle f-g, B(f-g) \rangle| \quad (\star)$$

$$\text{Recall that for } \forall f \in M, |\langle f, Bf \rangle| \leq T \cdot \|f\|^2.$$

$$\text{so } (\star) \leq T \cdot \|f+g\|^2 + T \cdot \|f-g\|^2$$

$$= T \cdot (\|f\|^2 + \langle f, g \rangle + \langle g, f \rangle + \|g\|^2 + \|f\|^2 - \langle f, g \rangle - \langle g, f \rangle + \|g\|^2)$$

$$= T \cdot (2\|f\|^2 + 2\|g\|^2)$$

$$\text{if } f, g \in M \text{ with } \|f\| = \|g\| = 1, \text{ then } (\star) = T \cdot 4$$

$$\text{so } \langle f, Bg \rangle \leq \frac{1}{4} \cdot T \cdot 4 = T. \text{ i.e. } T \geq \|B\|. \quad \textcircled{2}$$

$$\text{Thus by } \textcircled{1}\textcircled{2} \text{ we can obtain that } \|B\| = \sup_{f \in M, \|f\|=1} |\langle f, Bf \rangle|.$$

Extension:

Let  $P = \sum_{|\alpha| \leq m_1} A_\alpha(x) D_x^\alpha$  and  $Q = \sum_{|\beta| \leq m_2} B_\beta(x) D_x^\beta$  be differential operators on  $\mathbb{R}^n$ , where  $\alpha, \beta \in \mathbb{N}^n$  are multi-indices.  $A_\alpha(x), B_\beta(x) \in C_c^\infty(\mathbb{R}^n)$  and

$$D_x^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}, \quad D_x^\beta = (-i)^{|\beta|} \partial_x^\beta$$

(a) Show that for any  $f \in \mathcal{S}(\mathbb{R}^n)$  we can write

$$(Pf)(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix \cdot \xi} p(x, \xi) \hat{f}(\xi) d\xi, \text{ where } p(x, \xi) = \sum_{|\alpha| \leq m_1} A_\alpha(x) \xi^\alpha.$$

(b) Show that the composition  $PQ$  also can be written in this form with symbol

$$P_{PQ}(x, \xi) = \sum_{|\gamma| \leq m_1} \frac{1}{\gamma!} (\partial_\xi^\gamma p(x, \xi)) (D_x^\gamma q(x, \xi))$$

where  $q(x, \xi) = \sum_{|\beta| \leq m_2} B_\beta(x) \xi^\beta$  is the symbol of  $Q$ .

For this question, you will need the multi-index product rule.

$$D_x^\alpha (f(x)g(x)) = \sum_{\gamma=\delta+\beta} \binom{\alpha}{\gamma} (D_x^\delta f(x))(D_x^\beta g(x))$$

and the equation  $\xi^\gamma = \frac{(\alpha-\gamma)!}{\alpha!} \partial_\xi^\gamma (\xi^\alpha)$  for  $\gamma + \delta = \alpha$

where  $\alpha, \beta, \gamma \in \mathbb{N}^n$  are multi-indices.

(c) Show that  $PQ$  has order  $m_1 + m_2$  i.e.

$$|D_x^\alpha D_\xi^\beta P_{PQ}(x, \xi)| \leq C_{\alpha, \beta} (1+|\xi|)^{m_1+m_2-|\beta|}$$

Solutions:

(a) For any  $f \in \mathcal{S}(\mathbb{R}^n)$

$$(Pf)(x) = \left[ \sum_{|\alpha| \leq m_1} A_\alpha(x) D_x^\alpha f \right](x) = \sum_{|\alpha| \leq m_1} A_\alpha(x) [D_x^\alpha f](x)$$

$$\text{where } [D_x^\alpha f](x) = [\mathcal{F}^* \mathcal{F} D_x^\alpha f](x)$$

$$= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix \cdot \xi} [\mathcal{F} D_x^\alpha f](\xi) d\xi$$

$$= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \xi^\alpha [\mathcal{F} f](\xi) d\xi$$

$$= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \xi^\alpha \hat{f}(\xi) d\xi$$

$$\text{thus we get } (Pf)(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \sum_{|\alpha| \leq m_1} A_\alpha(x) \xi^\alpha \hat{f}(\xi) d\xi$$

$$= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix \cdot \xi} p(x, \xi) \hat{f}(\xi) d\xi$$

$$\text{with } p(x, \xi) = \sum_{|\alpha| \leq m_1} A_\alpha(x) \xi^\alpha$$

(b) On one hand

$$[PQf](x) = \sum_{|\alpha| \leq m_1} A_\alpha(x) [D_x^\alpha (\alpha f)](x)$$

$$\text{where } [D_x^\alpha (\alpha f)](x) = (-i)^{|\alpha|} \partial_x^\alpha \left[ \sum_{|\beta| \leq m_2} B_\beta(x) D_x^\beta f \right](x)$$

$$= (-i)^{|\alpha|} \sum_{|\beta| \leq m_2} \partial_x^\alpha \left[ B_\beta(x) D_x^\beta f \right](x)$$

$$= (-i)^{|\alpha|} \sum_{|\beta| \leq m_2} \left[ \sum_{|\gamma| \leq \alpha} \binom{\alpha}{\gamma} [\partial_x^\gamma B_\beta](x) \cdot (-i)^{|\beta|} \partial_x^{\alpha-\gamma} f(x) \right]$$

where  $\alpha = \gamma + \beta$ .

then we get

$$[PQf](x) = \sum_{|\alpha| \leq m_1} A_\alpha(x) (-i)^{|\alpha|} \sum_{|\beta| \leq m_2} \sum_{|\gamma| \leq \alpha} \binom{\alpha}{\gamma} [\partial_x^\gamma B_\beta](x) (-i)^{|\beta|} [\partial_x^{\alpha-\beta} f](x)$$

$$= \sum_{|\alpha| \leq m_1} A_\alpha(x) \sum_{|\beta| \leq m_2} \sum_{|\gamma| \leq \alpha} \binom{\alpha}{\gamma} (-i)^{|\alpha|+|\beta|} [\partial_x^\gamma B_\beta](x) [\partial_x^{\alpha-\beta} f](x)$$

$$= \sum_{|\alpha| \leq m_1} A_\alpha(x) \sum_{|\beta| \leq m_2} \sum_{|\gamma| \leq \alpha} \binom{\alpha}{\gamma} (-i)^{|\alpha|+|\beta|} (-i)^{-|\beta|+|\gamma|} [\partial_x^\gamma B_\beta](x) \cdot (-i)^{|\beta|+|\gamma|} [\partial_x^{\beta+\gamma} f](x)$$

$$= \sum_{|\alpha| \leq m_1} A_\alpha(x) \sum_{|\beta| \leq m_2} \sum_{|\gamma| \leq \alpha} \binom{\alpha}{\gamma} (-i)^{|\alpha|} [\partial_x^\gamma B_\beta](x) [\partial_x^{\beta+\gamma} f](x)$$

"M(x)

$$= M(x) [\mathcal{F}^* \mathcal{F} D_x^{\beta+\gamma} f](x)$$

$$= \frac{M(x)}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix \cdot \xi} [\mathcal{F} D_x^{\beta+\gamma} f](\xi) d\xi$$

$$= \frac{M(x)}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \xi^{\beta+\gamma} \hat{f}(\xi) d\xi \quad (*)$$

On the other hand:

$$\begin{aligned}
 P_{pq}(x, \xi) &= \sum_{|\gamma| \leq d} \frac{1}{\gamma!} \left( \partial_x^\gamma p(x, \xi) \right) \left( \partial_x^{-\gamma} q(x, \xi) \right) \\
 &= \sum_{|\gamma| \leq d} \frac{1}{\gamma!} \left[ \partial_x^\gamma \left( \sum_{|\alpha| \leq m_1} A_\alpha(x) \xi^\alpha \right) \right] \cdot (-i)^{|\gamma|} \partial_x^{-\gamma} \left( \sum_{|\beta| \leq m_2} B_\beta(x) \xi^\beta \right) \\
 &= \sum_{|\gamma| \leq d} \frac{1}{\gamma!} \sum_{|\alpha| \leq m_1} A_\alpha(x) \partial_x^\gamma (\xi^\alpha) \cdot (-i)^{|\gamma|} \sum_{|\beta| \leq m_2} [\partial_x^{-\gamma} B_\beta](x) \xi^\beta \\
 &= \sum_{|\gamma| \leq d} \frac{1}{\gamma!} \sum_{|\alpha| \leq m_1} A_\alpha(x) \frac{\alpha!}{(d-\gamma)!} \xi^\gamma \cdot (-i)^{|\gamma|} \sum_{|\beta| \leq m_2} [\partial_x^{-\gamma} B_\beta](x) \xi^\beta \\
 &= \sum_{|\gamma| \leq d} \binom{d}{\gamma} \sum_{|\alpha| \leq m_1} A_\alpha(x) (-i)^{|\gamma|} \sum_{|\beta| \leq m_2} (\partial_x^{-\gamma} B_\beta)(x) \xi^{\gamma + \beta} \\
 &\quad \text{also equals to } M(x) \\
 &= M(x) \xi^{\beta + \gamma}
 \end{aligned}$$

then by (a), we have

$$\begin{aligned}
 [PQf](x) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix \cdot \xi} P_{pq}(x, \xi) \hat{f}(\xi) d\xi \\
 &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix \cdot \xi} M(x) \xi^{\beta + \gamma} \hat{f}(\xi) d\xi \\
 &= \frac{M(x)}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \xi^{\beta + \gamma} \hat{f}(\xi) d\xi \quad (**)
 \end{aligned}$$

We can see that (\*) and (\*\*) are the same.

so PQ can be written in this form with symbol  $P_{pq}(x, \xi)$ .

(c). In this section, we let  $q(x, \xi) = \sum_{|\beta| \leq m_2} B_\beta(x) \xi^\beta$   
 $p(x, \xi) = \sum_{|\alpha| \leq m_1} A_\alpha(x) \xi^\alpha$   
and  $\beta = \eta + \lambda$        $\alpha = \gamma + \delta$ .

then consider  $\hat{P}_q^\beta P_{pq}(x, \xi)$ .

$$D_{\beta}^{\beta} P_{\beta}(x, \xi) = (-i)^{|\beta|} \partial_{\xi}^{\beta} \left[ \left( \sum_{|\alpha| \leq d_1} \frac{1}{\alpha!} \partial_x^{\alpha} \left( \sum_{|\beta'| \leq m_1} A_{\alpha'}(x) \xi^{\beta'} \right) \right) \left( (-i)^{|\gamma|} \partial_x^{\gamma} \sum_{|\beta'| \leq m_2} B_{\beta'}(x) \xi^{\beta'} \right) \right]$$

$$= (-i)^{|\beta|+|\gamma|} \underbrace{\sum_{|\alpha| \leq d_1} \frac{1}{\alpha!} \sum_{|\beta'| \leq m_1} A_{\alpha'}(x) \sum_{|\beta'| \leq m_2} B_{\beta'}(x)}_{=: N(x)} \partial_{\xi}^{\beta} \left[ (\partial_{\xi}^{\gamma} (\xi^{\alpha})) \cdot (\partial_x^{\gamma} (\xi^{\beta})) \right]$$

$$= N(x) \sum_{|\eta| \leq \beta} \binom{\beta}{\eta} \partial_{\xi}^{\gamma+\eta} (\xi^{\alpha}) \partial_x^{\gamma+\beta-\eta} (\xi^{\beta})$$

$$= N(x) \sum_{|\eta| \leq \beta} \binom{\beta}{\eta} \frac{\alpha'!}{(\alpha'-\gamma-\eta)!} \xi^{\alpha'-\gamma-\eta} \cdot \frac{\beta'!}{(\beta'-\gamma-\beta+\eta)!} \xi^{\beta'-\gamma-\beta+\eta}$$

$$= N(x) \sum_{|\eta| \leq \beta} \binom{\beta}{\eta} \frac{\alpha'! \beta'!}{(\alpha'-\gamma-\eta)! (\beta'-\gamma+\eta-\beta)!} \xi^{\alpha'+\beta'-2\gamma-\beta}$$

Since  $|\alpha'| \leq m_1$ ,  $|\beta'| \leq m_2$ ,  $0 \leq |\gamma| \leq |\alpha'|$

so the maximal order of  $\xi$  could be  $m_1 + m_2 - |\beta|$

Since  $D_x^{\alpha} [D_{\beta}^{\beta} P_{\beta}(x, \xi)]$  has nothing to do with the order of  $\xi$  and  $A_{\alpha}, B_{\beta} \in C_c^{\infty}(\mathbb{R}^n)$  s.t.

$D_x^{\alpha} [D_{\beta}^{\beta} P_{\beta}(x, \xi)]$  can be bounded by  $C_{\alpha, \beta} (1+|\beta|)^{m_1+m_2-|\beta|}$ .