Chapter 1

Hilbert space and bounded linear operators

This chapter is mainly based on the first two chapters of the book [Amr]. Its content is quite standard and this theory can be seen as a special instance of bounded linear operators on more general Banach spaces.

1.1 Hilbert space

Definition 1.1.1. A (complex) Hilbert space $\mathcal{H}$ *is a vector space on* $\mathbb{C}$ *with a strictly positive scalar product (or inner product) which is complete for the associated norm*[1] *and which admits a countable orthonormal basis. The scalar product is denoted by* $\langle \cdot, \cdot \rangle$ *and the corresponding norm by* $\| \cdot \|$.

In particular, note that for any $f, g, h \in \mathcal{H}$ and $\alpha \in \mathbb{C}$ the following properties hold:

(i) $\langle f, g \rangle = \overline{\langle g, f \rangle}$,

(ii) $\langle f, g + \alpha h \rangle = \langle f, g \rangle + \alpha \langle f, h \rangle$,

(iii) $\|f\|^2 = \langle f, f \rangle \geq 0$, and $\|f\| = 0$ if and only if $f = 0$.

Note that $\overline{\langle g, f \rangle}$ means the complex conjugate of $\langle g, f \rangle$. Note also that the linearity in the second argument in (ii) is a matter of convention, many authors define the linearity in the first argument. In (iii) the norm of f is defined in terms of the scalar product $\langle f, f \rangle$. We emphasize that the scalar product can also be defined in terms of the norm of $\mathcal{H}$, this is the content of the *polarisation identity*:

$$4\langle f, g \rangle = \|f + g\|^2 - \|f - g\|^2 - i\|f + ig\|^2 + i\|f - ig\|^2. \qquad (1.1)$$

[1] Recall that $\mathcal{H}$ is said to be complete if any Cauchy sequence in $\mathcal{H}$ has a limit in $\mathcal{H}$. More precisely, $\{f_n\}_{n \in \mathbb{N}} \subset \mathcal{H}$ is a Cauchy sequence if for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $\|f_n - f_m\| < \varepsilon$ for any $n, m \geq N$. Then $\mathcal{H}$ is complete if for any such sequence there exists $f_\infty \in \mathcal{H}$ such that $\lim_{n \to \infty} \|f_n - f_\infty\| = 0$.

From now on, the symbol $\mathcal{H}$ will always denote a Hilbert space.

Examples 1.1.2. *(i)* $\mathcal{H} = \mathbb{C}^d$ *with* $\langle \alpha, \beta\rangle = \sum_{j=1}^d \overline{\alpha_j}\beta_j$ *for any* $\alpha, \beta \in \mathbb{C}^d$,

(ii) $\mathcal{H} = l^2(\mathbb{Z})$ *with* $\langle a, b\rangle = \sum_{j\in\mathbb{Z}} \overline{a_j} b_j$ *for any* $a, b \in l^2(\mathbb{Z})$,

(iii) $\mathcal{H} = L^2(\mathbb{R}^d)$ *with* $\langle f, g\rangle = \int_{\mathbb{R}^d} \overline{f(x)}\, g(x)\,\mathrm{d}x$ *for any* $f, g \in L^2(\mathbb{R}^d)$.

Let us recall some useful inequalities: For any $f, g \in \mathcal{H}$ one has

$$|\langle f, g\rangle| \leq \|f\|\,\|g\| \qquad \text{Schwarz inequality,} \tag{1.2}$$

$$\|f+g\| \leq \|f\| + \|g\| \qquad \text{triangle inequality,} \tag{1.3}$$

$$\|f+g\|^2 \leq 2\|f\|^2 + 2\|g\|^2, \tag{1.4}$$

$$\big|\|f\| - \|g\|\big| \leq \|f-g\|. \tag{1.5}$$

The proof of these inequalities is standard and is left as a free exercise, see also [Amr, p. 3-4]. Let us also recall that $f, g \in \mathcal{H}$ are said to be *orthogonal* if $\langle f, g\rangle = 0$.

Definition 1.1.3. *A sequence* $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$ *is* strongly convergent *to* $f_\infty \in \mathcal{H}$ *if* $\lim_{n\to\infty}\|f_n - f_\infty\| = 0$*, or is* weakly convergent *to* $f_\infty \in \mathcal{H}$ *if for any* $g \in \mathcal{H}$ *one has* $\lim_{n\to\infty}\langle g, f_n - f_\infty\rangle = 0$*. One writes* $\mathrm{s-}\lim_{n\to\infty} f_n = f_\infty$ *if the sequence is strongly convergent, and* $\mathrm{w-}\lim_{n\to\infty} f_n = f_\infty$ *if the sequence is weakly convergent.*

Clearly, a strongly convergent sequence is also weakly convergent. The converse is not true.

Exercise 1.1.4. *In the Hilbert space* $L^2(\mathbb{R})$*, exhibit a sequence which is weakly convergent but not strongly convergent.*

Lemma 1.1.5. *Consider a sequence* $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$*. One has*

$$\mathrm{s-}\lim_{n\to\infty} f_n = f_\infty \quad\Longleftrightarrow\quad \mathrm{w-}\lim_{n\to\infty} f_n = f_\infty \ \ \textit{and} \ \ \lim_{n\to\infty}\|f_n\| = \|f_\infty\|.$$

Proof. Assume first that $\mathrm{s-}\lim_{n\to\infty} f_n = f_\infty$. By the Schwarz inequality one infers that for any $g \in \mathcal{H}$:

$$|\langle g, f_n - f_\infty\rangle| \leq \|f_n - f_\infty\|\,\|g\| \to 0 \qquad \text{as } n\to\infty,$$

which means that $\mathrm{w-}\lim_{n\to\infty} f_n = f_\infty$. In addition, by (1.5) one also gets

$$\big|\|f_n\| - \|f_\infty\|\big| \leq \|f_n - f_\infty\| \to 0 \quad \text{as} \quad n\to\infty,$$

and thus $\lim_{n\to\infty}\|f_n\| = \|f_\infty\|$.

For the reverse implication, observe first that

$$\|f_n - f_\infty\|^2 = \|f_n\|^2 + \|f_\infty\|^2 - \langle f_n, f_\infty\rangle - \langle f_\infty, f_n\rangle. \tag{1.6}$$

If $\mathrm{w-}\lim_{n\to\infty} f_n = f_\infty$ and $\lim_{n\to\infty}\|f_n\| = \|f_\infty\|$, then the right-hand side of (1.6) converges to $\|f_\infty\|^2 + \|f_\infty\|^2 - \|f_\infty\|^2 - \|f_\infty\|^2 = 0$, so that $\mathrm{s-}\lim_{n\to\infty} f_n = f_\infty$. □

Let us also note that if $\mathrm{s-}\lim_{n\to\infty} f_n = f_\infty$ and $\mathrm{s-}\lim_{n\to\infty} g_n = g_\infty$ then one has

$$\lim_{n\to\infty} \langle f_n, g_n\rangle = \langle f_\infty, g_\infty\rangle$$

by a simple application of the Schwarz inequality.

Exercise 1.1.6. *Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis of an infinite dimensional Hilbert space. Show that $\mathrm{w-}\lim_{n\to\infty} e_n = 0$, but that $\mathrm{s-}\lim_{n\to\infty} e_n$ does not exist.*

Exercise 1.1.7. *Show that the limit of a strong or a weak Cauchy sequence is unique. Show also that such a sequence is bounded,* i.e. *if $\{f_n\}_{n\in\mathbb{N}}$ denotes this Cauchy sequence, then $\sup_{n\in\mathbb{N}} \|f_n\| < \infty$.*

For the weak Cauchy sequence, the boundedness can be obtained from the following quite general result which will be useful later on. Its proof can be found in [Kat, Thm. III.1.29]. In the statement, Λ is simply a set.

Theorem 1.1.8 (Uniform boundedness principle)**.** *Let $\{\varphi_\lambda\}_{\lambda\in\Lambda}$ be a family of continuous maps*[2] *$\varphi_\lambda : \mathcal{H} \to [0,\infty)$ satisfying*

$$\varphi_\lambda(f+g) \leq \varphi_\lambda(f) + \varphi_\lambda(g) \qquad \forall f, g \in \mathcal{H}.$$

If the set $\{\varphi_\lambda(f)\}_{\lambda\in\Lambda} \subset [0,\infty)$ is bounded for any fixed $f \in \mathcal{H}$, then the family $\{\varphi_\lambda\}_{\lambda\in\Lambda}$ is uniformly bounded, i.e. *there exists $c > 0$ such that $\sup_\lambda \varphi_\lambda(f) \leq c$ for any $f \in \mathcal{H}$ with $\|f\| = 1$.*

In the next definition, we introduce the notion of a linear manifold and of a subspace of a Hilbert space.

Definition 1.1.9. A linear manifold *$\mathcal{M}$ of a Hilbert space $\mathcal{H}$ is a linear subset of $\mathcal{H}$, or more precisely $\forall f, g \in \mathcal{M}$ and $\alpha \in \mathbb{C}$ one has $f + \alpha g \in \mathcal{M}$. If $\mathcal{M}$ is closed ($\Leftrightarrow$ any Cauchy sequence in $\mathcal{M}$ converges strongly in $\mathcal{M}$), then $\mathcal{M}$ is called a* subspace *of $\mathcal{H}$.*

Note that if $\mathcal{M}$ is closed, then $\mathcal{M}$ is a Hilbert space in itself, with the scalar product and norm inherited from $\mathcal{H}$. Be aware that some authors call *subspace* what we have defined as a linear manifold, and call *closed subspace* what we simply call a subspace.

Examples 1.1.10. *(i) If $f_1, \dots, f_n \in \mathcal{H}$, then $\mathsf{Vect}(f_1, \dots, f_n)$ is the closed vector space generated by the linear combinations of $f_1, \dots f_n$. $\mathsf{Vect}(f_1, \dots, f_n)$ is a subspace.*

(ii) If $\mathcal{M}$ is a subset of $\mathcal{H}$, then

$$\mathcal{M}^\perp := \{f \in \mathcal{H} \mid \langle f, g\rangle = 0, \forall g \in \mathcal{M}\} \tag{1.7}$$

is a subspace of $\mathcal{H}$.

[2] φ_λ is continuous if $\varphi_\lambda(f_n) \to \varphi_\lambda(f_\infty)$ whenever $\mathrm{s-}\lim_{n\to\infty} f_n = f_\infty$.

Exercise 1.1.11. *Check that in the above example the set* $\mathcal{M}^\perp$ *is a subspace of* $\mathcal{H}$.

Exercise 1.1.12. *Check that a linear manifold* $\mathcal{M} \subset \mathcal{H}$ *is dense in* $\mathcal{H}$ *if and only if* $\mathcal{M}^\perp = \{0\}$.

If $\mathcal{M}$ is a subset of $\mathcal{H}$ the subspace $\mathcal{M}^\perp$ is called *the orthocomplement of* $\mathcal{M}$ *in* $\mathcal{H}$. The following result is important in the setting of Hilbert spaces. Its proof is not complicated but a little bit lengthy, we thus refer to [Amr, Prop. 1.7].

Proposition 1.1.13 (Projection Theorem)**.** *Let* $\mathcal{M}$ *be a subspace of a Hilbert space* $\mathcal{H}$. *Then, for any* $f \in \mathcal{H}$ *there exist a unique* $f_1 \in \mathcal{M}$ *and a unique* $f_2 \in \mathcal{M}^\perp$ *such that* $f = f_1 + f_2$.

Let us close this section with the so-called Riesz Lemma. For that purpose, recall first that the dual $\mathcal{H}^*$ of the Hilbert space $\mathcal{H}$ consists in the set of all bounded linear functionals on $\mathcal{H}$, *i.e.* $\mathcal{H}^*$ consists in all mappings $\varphi : \mathcal{H} \to \mathbb{C}$ satisfying for any $f, g \in \mathcal{H}$ and $\alpha \in \mathbb{C}$

(i) $\varphi(f + \alpha g) = \varphi(f) + \alpha\varphi(g)$, (linearity)

(ii) $|\varphi(f)| \leq c\|f\|$, (boundedness)

where c is a constant independent of f. One then sets

$$\|\varphi\|_{\mathcal{H}^*} := \sup_{0 \neq f \in \mathcal{H}} \frac{|\varphi(f)|}{\|f\|}.$$

Clearly, if $g \in \mathcal{H}$, then g defines an element φ_g of $\mathcal{H}^*$ by setting $\varphi_g(f) := \langle g, f\rangle$. Indeed φ_g is linear and one has

$$\|\varphi_g\|_{\mathcal{H}^*} := \sup_{0 \neq f \in \mathcal{H}} \frac{1}{\|f\|}|\langle g, f\rangle| \leq \sup_{0 \neq f \in \mathcal{H}} \frac{1}{\|f\|}\|g\|\,\|f\| = \|g\|.$$

In fact, note that $\|\varphi_g\|_{\mathcal{H}^*} = \|g\|$ since $\frac{1}{\|g\|}\varphi_g(g) = \frac{1}{\|g\|}\|g\|^2 = \|g\|$.

The following statement shows that any element $\varphi \in \mathcal{H}^*$ can be obtained from an element $g \in \mathcal{H}$. It corresponds thus to a converse of the previous construction.

Lemma 1.1.14 (Riesz Lemma)**.** *For any* $\varphi \in \mathcal{H}^*$, *there exists a unique* $g \in \mathcal{H}$ *such that for any* $f \in \mathcal{H}$

$$\varphi(f) = \langle g, f\rangle.$$

In addition, g *satisfies* $\|\varphi\|_{\mathcal{H}^*} = \|g\|$.

Since the proof is quite standard, we only sketch it and leave the details to the reader, see also [Amr, Prop. 1.8].

Sketch of the proof. If $\varphi \equiv 0$, then one can set $g := 0$ and observe trivially that $\varphi = \varphi_g$.

If $\varphi \neq 0$, let us first define $\mathcal{M} := \{f \in \mathcal{H} \mid \varphi(f) = 0\}$ and observe that $\mathcal{M}$ is a subspace of $\mathcal{H}$. One also observes that $\mathcal{M} \neq \mathcal{H}$ since otherwise $\varphi \equiv 0$. Thus, let $h \in \mathcal{H}$ such that $\varphi(h) \neq 0$ and decompose $h = h_1 + h_2$ with $h_1 \in \mathcal{M}$ and $h_2 \in \mathcal{M}^\perp$ by Proposition 1.1.13. One infers then that $\varphi(h_2) = \varphi(h) \neq 0$.

For arbitrary $f \in \mathcal{H}$ one can consider the element $f - \frac{\varphi(f)}{\varphi(h_2)} h_2 \in \mathcal{H}$ and observe that $\varphi\big(f - \frac{\varphi(f)}{\varphi(h_2)} h_2\big) = 0$. One deduces that $f - \frac{\varphi(f)}{\varphi(h_2)} h_2$ belongs to $\mathcal{M}$, and since $h_2 \in \mathcal{M}^\perp$ one infers that

$$\varphi(f) = \frac{\varphi(h_2)}{\|h_2\|^2} \langle h_2, f\rangle.$$

One can thus set $g := \overline{\frac{\varphi(h_2)}{\|h_2\|^2}} h_2 \in \mathcal{H}$ and easily obtain the remaining parts of the statement. □

As a consequence of the previous statement, one often identifies $\mathcal{H}^*$ with $\mathcal{H}$ itself.

Exercise 1.1.15. *Check that this identification is not linear but* anti-linear*.*

1.2 Vector-valued functions

Let $\mathcal{H}$ be a Hilbert space and let Λ be a set. A *vector-valued function* if a map $f : \Lambda \to \mathcal{H}$, *i.e.* for any $\lambda \in \Lambda$ one has $f(\lambda) \in \mathcal{H}$. In application, we shall mostly consider the special case $\Lambda = \mathbb{R}$ or $\Lambda = [a, b]$ with $a, b \in \mathbb{R}$ and $a < b$.

The following definitions are mimicked from the special case $\mathcal{H} = \mathbb{C}$, but different topologies on $\mathcal{H}$ can be considered:

Definition 1.2.1. *Let $J := (a, b)$ with $a < b$ and consider a vector-valued function $f : J \to \mathcal{H}$.*

(i) f is strongly continuous *on J if for any $t \in J$ one has* $\lim_{\varepsilon\to 0} \|f(t+\varepsilon) - f(t)\| = 0$,

(ii) f is weakly continuous *on J if for any $t \in J$ and any $g \in \mathcal{H}$ one has*

$$\lim_{\varepsilon\to 0} \big\langle g, f(t+\varepsilon) - f(t)\big\rangle = 0,$$

(iii) f is strongly differentiable *on J if there exists another vector-valued function $f' : J \to \mathcal{H}$ such that for any $t \in J$ one has*

$$\lim_{\varepsilon\to 0} \big\| \tfrac{1}{\varepsilon}\big(f(t+\varepsilon) - f(t)\big) - f'(t)\big\| = 0,$$

(iii) f is weakly differentiable *on J if there exists another vector-valued function $f' : J \to \mathcal{H}$ such that for any $t \in J$ and $g \in \mathcal{H}$ one has*

$$\lim_{\varepsilon\to 0} \big\langle g, \tfrac{1}{\varepsilon}\big(f(t+\varepsilon) - f(t)\big) - f'(t)\big\rangle = 0,$$

The map f' is called the strong derivative*, respectively the* weak derivative*, of f.*

Integrals of vector-valued functions can be defined in several senses, but we shall restrict ourselves to Riemann-type integrals. The construction is then similar to real or complex-valued functions, by considering finer and finer partitions of a bounded interval J. Improper Riemann integrals can also be defined in analogy with the scalar case by a limiting process. Note that these integrals can exist either in the strong sense (strong topology on $\mathcal{H}$) or in the weak sense (weak topology on $\mathcal{H}$). In the sequel, we consider only the existence of such integrals in the strong sense.

Let us thus consider $J := (a, b]$ with $a < b$ and let us set $\Pi = \{s_0, \dots, s_n; u_1, \dots u_n\}$ with $a = s_0 < u_1 \leq s_1 < u_2 \leq s_2 < \cdots < u_n \leq s_n = b$ for a partition of J. One also sets $|\Pi| := \max_{k\in\{1,\dots,n\}} |s_k - s_{k-1}|$ and the Riemann sum

$$\Sigma(\Pi, f) := \sum_{k=1}^{n} (s_k - s_{k-1}) f(u_k).$$

If one considers then a sequence $\{\Pi_i\}_{i\in\mathbb{N}}$ of partitions of J with $|\Pi_i| \to 0$ as $i \to \infty$ one writes

$$\int_J f(t)\mathrm{d}t \equiv \int_a^b f(t)\mathrm{d}t = \mathrm{s}-\lim_{i\to\infty} \Sigma(\Pi_i, f)$$

if this limit exists and is independent of the sequence of partitions. In this case, one says that f is *strongly integrable* on $(a, b]$. Clearly, similar definitions hold for $J = (a, b)$ or $J = [a, b]$. Infinite intervals can be considered by a limiting process as long as the corresponding limits exist.

The following statements can then be proved in a way similar to the scalar case.

Proposition 1.2.2. *Let $(a, b]$ and $(b, c]$ be finite or infinite intervals and suppose that all the subsequent integrals exist. Then one has*

(i) $\int_a^b f(t)\mathrm{d}t + \int_b^c f(t)\mathrm{d}t = \int_a^c f(t)\mathrm{d}t$,

(ii) $\int_a^b \big(\alpha f_1(t) + f_2(t)\big)\mathrm{d}t = \alpha \int_a^b f_1(t)\mathrm{d}t + \int_a^b f_2(t)\mathrm{d}t$,

(iii) $\Big\| \int_a^b f(t)\mathrm{d}t \Big\| \leq \int_a^b \|f(t)\|\mathrm{d}t$.

For the existence of these integrals one has:

Proposition 1.2.3. *(i) If $[a, b]$ is a finite closed interval and $f : [a, b] \to \mathcal{H}$ is strongly continuous, then $\int_a^b f(t)\mathrm{d}t$ exists,*

(ii) If $a < b$ are arbitrary and $\int_a^b \|f(t)\|\mathrm{d}t < \infty$, then $\int_a^b f(t)\mathrm{d}t$ exists,

(iii) If f is strongly differentiable on (a, b) and its derivative f' is strongly continuous and integrable on $[a, b]$ then

$$\int_a^b f'(t)\mathrm{d}t = f(b) - f(a).$$

1.3 Bounded linear operators

First of all, let us recall that a linear map B between two complex vector spaces $\mathcal{M}$ and $\mathcal{N}$ satisfies $B(f+\alpha g) = Bf + \alpha Bg$ for all $f, g \in \mathcal{M}$ and $\alpha \in \mathbb{C}$.

Definition 1.3.1. *A map $B : \mathcal{H} \to \mathcal{H}$ is* a bounded linear operator *if $B : \mathcal{H} \to \mathcal{H}$ is a linear map, and if there exists $c > 0$ such that $\|Bf\| \leq c\|f\|$ for all $f \in \mathcal{H}$. The set of* all bounded linear operators on $\mathcal{H}$ *is denoted by $\mathscr{B}(\mathcal{H})$.*

For any $B \in \mathscr{B}(\mathcal{H})$, one sets

$$\begin{aligned} \|B\| &:= \inf\{c > 0 \mid \|Bf\| \leq c\|f\| \ \forall f \in \mathcal{H}\} \\ &= \sup_{0 \neq f \in \mathcal{H}} \frac{\|Bf\|}{\|f\|}. \end{aligned} \tag{1.8}$$

and call it *the norm of B*. Note that the same notation is used for the norm of an element of $\mathcal{H}$ and for the norm of an element of $\mathscr{B}(\mathcal{H})$, but this does not lead to any confusion. Let us also introduce the *range* of an operator $B \in \mathscr{B}(\mathcal{H})$, namely

$$\mathsf{Ran}(B) := B\mathcal{H} = \{f \in \mathcal{H} \mid f = Bg \text{ for some } g \in \mathcal{H}\}. \tag{1.9}$$

This notion will be important when the inverse of an operator will be discussed.

Exercise 1.3.2. *Let $\mathcal{M}_1, \mathcal{M}_2$ be two dense linear manifolds of $\mathcal{H}$, and let $B \in \mathscr{B}(\mathcal{H})$. Show that*

$$\|B\| = \sup_{f \in \mathcal{M}_1, g \in \mathcal{M}_2 \text{ with } \|f\| = \|g\| = 1} |\langle f, Bg\rangle|. \tag{1.10}$$

Exercise 1.3.3. *Show that $\mathscr{B}(\mathcal{H})$ is a complete normed algebra and that the inequality*

$$\|AB\| \leq \|A\| \, \|B\| \tag{1.11}$$

holds for any $A, B \in \mathscr{B}(\mathcal{H})$.

An additional structure can be added to $\mathscr{B}(\mathcal{H})$: an involution. More precisely, for any $B \in \mathscr{B}(\mathcal{H})$ and any $f, g \in \mathcal{H}$ one sets

$$\langle B^* f, g\rangle := \langle f, Bg\rangle. \tag{1.12}$$

Exercise 1.3.4. *For any $B \in \mathscr{B}(\mathcal{H})$ show that*

(i) B^* *is uniquely defined by* (1.12) *and satisfies* $B^* \in \mathscr{B}(\mathcal{H})$ *with* $\|B^*\| = \|B\|$,

(ii) $(B^*)^* = B$,

(iii) $\|B^*B\| = \|B\|^2$,

(iv) *If* $A \in \mathscr{B}(\mathcal{H})$, *then* $(AB)^* = B^*A^*$.

The operator B^* is called *the adjoint of* B, and the proof the unicity in (i) involves the Riesz Lemma. A complete normed algebra endowed with an involution for which the property (iii) holds is called a C^*-algebra. In particular $\mathscr{B}(\mathcal{H})$ is a C^*-algebra. Such algebras have a well-developed and deep theory, see for example [Mur]. However, we shall not go further in this direction in this course.

We have already considered two distinct topologies on $\mathcal{H}$, namely the strong and the weak topology. On $\mathscr{B}(\mathcal{H})$ there exist several topologies, but we shall consider only three of them.

Definition 1.3.5. *A sequence $\{B_n\}_{n\in\mathbb{N}} \subset \mathscr{B}(\mathcal{H})$ is* uniformly convergent *to $B_\infty \in \mathscr{B}(\mathcal{H})$ if $\lim_{n\to\infty} \|B_n - B_\infty\| = 0$, is* strongly convergent *to $B_\infty \in \mathscr{B}(\mathcal{H})$ if for any $f \in \mathcal{H}$ one has $\lim_{n\to\infty} \|B_n f - B_\infty f\| = 0$, or is* weakly convergent *to $B_\infty \in \mathscr{B}(\mathcal{H})$ if for any $f, g \in \mathcal{H}$ one has $\lim_{n\to\infty}\langle f, B_n g - B_\infty g\rangle = 0$. In these cases, one writes respectively $u - \lim_{n\to\infty} B_n = B_\infty$, $s - \lim_{n\to\infty} B_n = B_\infty$ and $w - \lim_{n\to\infty} B_n = B_\infty$.*

Clearly, uniform convergence implies strong convergence, and strong convergence implies weak convergence. The reverse statements are not true. Note that if $\{B_n\}_{n\in\mathbb{N}} \subset \mathscr{B}(\mathcal{H})$ is weakly convergent, then the sequence $\{B_n^*\}_{n\in\mathbb{N}}$ of its adjoint operators is also weakly convergent. However, the same statement does not hold for a strongly convergent sequence. Finally, we shall not prove but often use that $\mathscr{B}(\mathcal{H})$ is also weakly and strongly closed. In other words, any weakly (or strongly) Cauchy sequence in $\mathscr{B}(\mathcal{H})$ converges in $\mathscr{B}(\mathcal{H})$.

Exercise 1.3.6. *Let $\{A_n\}_{n\in\mathbb{N}} \subset \mathscr{B}(\mathcal{H})$ and $\{B_n\}_{n\in\mathbb{N}} \subset \mathscr{B}(\mathcal{H})$ be two strongly convergent sequence in $\mathscr{B}(\mathcal{H})$, with limits A_∞ and B_∞ respectively. Show that the sequence $\{A_nB_n\}_{n\in\mathbb{N}}$ is strongly convergent to the element $A_\infty B_\infty$.*

Let us close this section with some information about the inverse of a bounded operator. Additional information on the inverse in relation with unbounded operators will be provided in the sequel.

Definition 1.3.7. *An operator $B \in \mathscr{B}(\mathcal{H})$ is* invertible *if the equation $Bf = 0$ only admits the solution $f = 0$. In such a case, there exists a linear map $B^{-1} : \mathsf{Ran}(B) \to \mathcal{H}$ which satisfies $B^{-1}Bf = f$ for any $f \in \mathcal{H}$, and $BB^{-1}g = g$ for any $g \in \mathsf{Ran}(B)$. If B is invertible and $\mathsf{Ran}(B) = \mathcal{H}$, then $B^{-1} \in \mathscr{B}(\mathcal{H})$ and B is said to be* invertible in $\mathscr{B}(\mathcal{H})$ *(or* boundedly invertible*).*

Note that the two conditions B invertible and $\mathsf{Ran}(B) = \mathcal{H}$ imply $B^{-1} \in \mathscr{B}(\mathcal{H})$ is a consequence of the Closed graph Theorem. In the sequel, we shall use the notation $\mathbf{1} \in \mathscr{B}(\mathcal{H})$ for the operator defined on any $f \in \mathcal{H}$ by $\mathbf{1}f = f$, and $\mathbf{0} \in \mathscr{B}(\mathcal{H})$ for the operator defined by $\mathbf{0}f = 0$.

The next statement is very useful in applications, and holds in a much more general context. Its proof is classical and can be found in every textbook.

Lemma 1.3.8 (Neumann series). *If $B \in \mathscr{B}(\mathcal{H})$ and $\|B\| < 1$, then the operator $(\mathbf{1}-B)$ is invertible in $\mathscr{B}(\mathcal{H})$, with*

$$(\mathbf{1} - B)^{-1} = \sum_{n=0}^{\infty} B^n,$$

and with $\|(\mathbf{1} - B)^{-1}\| \leq (1 - \|B\|)^{-1}$. The series converges in the uniform norm of $\mathscr{B}(\mathcal{H})$.

Note that we have used the identity $B^0 = \mathbf{1}$.

1.4 Special classes of bounded linear operators

In this section we provide some information on some subsets of $\mathscr{B}(\mathcal{H})$. We start with some operators which will play an important role in the sequel.

Definition 1.4.1. *An operator $B \in \mathscr{B}(\mathcal{H})$ is called* self-adjoint *if $B^* = B$, or equivalently if for any $f, g \in \mathcal{H}$ one has*

$$\langle f, Bg\rangle = \langle Bf, g\rangle. \tag{1.13}$$

For these operators the computation of their norm can be simplified (see also Exercise 1.3.2) :

Exercise 1.4.2. *If $B \in \mathscr{B}(\mathcal{H})$ is self-adjoint and if $\mathcal{M}$ is a dense linear manifold in $\mathcal{H}$, show that*

$$\|B\| = \sup_{f\in\mathcal{M},\ \|f\|=1} |\langle f, Bf\rangle|. \tag{1.14}$$

A special set of self-adjoint operators is provided by the set of orthogonal projections:

Definition 1.4.3. *An element $P \in \mathscr{B}(\mathcal{H})$ is* an orthogonal projection *if $P = P^2 = P^*$.*

It not difficult to check that there is a one-to-one correspondence between the set of subspaces of $\mathcal{H}$ and the set of orthogonal projections in $\mathscr{B}(\mathcal{H})$. Indeed, any orthogonal projection P defines a subspace $\mathcal{M} := P\mathcal{H}$. Conversely by taking the projection Theorem (Proposition 1.1.13) into account one infers that for any subspace $\mathcal{M}$ one can define an orthogonal projection P with $P\mathcal{H} = \mathcal{M}$.

In the sequel, we might simply say projection instead of orthogonal projection. However, let us stress that in other contexts a projection is often an operator P satisfying only the relation $P^2 = P$.

We gather in the next exercise some easy relations between orthogonal projections and the underlying subspaces. For that purpose we use the notation $P_\mathcal{M}, P_\mathcal{N}$ for the orthogonal projections on the subspaces $\mathcal{M}$ and $\mathcal{N}$ of $\mathcal{H}$.

Exercise 1.4.4. *Show the following relations:*

(i) *If* $P_{\mathcal{M}}P_{\mathcal{N}} = P_{\mathcal{N}}P_{\mathcal{M}}$, *then* $P_{\mathcal{M}}P_{\mathcal{N}}$ *is a projection and the associated subspace is* $\mathcal{M} \cap \mathcal{N}$,

(ii) *If* $\mathcal{M} \subset \mathcal{N}$, *then* $P_{\mathcal{M}}P_{\mathcal{N}} = P_{\mathcal{N}}P_{\mathcal{M}} = P_{\mathcal{M}}$,

(iii) *If* $\mathcal{M} \perp \mathcal{N}$, *then* $P_{\mathcal{M}}P_{\mathcal{N}} = P_{\mathcal{N}}P_{\mathcal{M}} = \mathbf{0}$, *and* $P_{\mathcal{M}\oplus\mathcal{N}} = P_{\mathcal{M}} + P_{\mathcal{N}}$,

(iv) *If* $P_{\mathcal{M}}P_{\mathcal{N}} = \mathbf{0}$, *then* $\mathcal{M} \perp \mathcal{N}$.

Let us now consider unitary operators, and then more general isometries and partial isometries. For that purpose, we recall that $\mathbf{1}$ denotes the identify operator in $\mathscr{B}(\mathcal{H})$.

Definition 1.4.5. *An element* $U \in \mathscr{B}(\mathcal{H})$ *is* a unitary operator *if* $UU^* = \mathbf{1}$ *and if* $U^*U = \mathbf{1}$.

Note that if U is unitary, then U is invertible in $\mathscr{B}(\mathcal{H})$ with $U^{-1} = U^*$. Indeed, observe first that $Uf = 0$ implies $f = U^*(Uf) = U^*0 = 0$. Secondly, for any $g \in \mathcal{H}$, one has $g = U(U^*g)$, and thus $\mathsf{Ran}(U) = \mathcal{H}$. Finally, the equality $U^{-1} = U^*$ follows from the unicity of the inverse.

More generally, an element $V \in \mathscr{B}(\mathcal{H})$ is called *an isometry* if the equality

$$V^*V = \mathbf{1} \tag{1.15}$$

holds. Clearly, a unitary operator is an instance of an isometry. For isometries the following properties can easily be obtained.

Proposition 1.4.6. *a) Let* $V \in \mathscr{B}(\mathcal{H})$ *be an isometry. Then*

(i) V *preserves the scalar product, namely* $\langle Vf, Vg\rangle = \langle f, g\rangle$ *for any* $f, g \in \mathcal{H}$,

(ii) V *preserves the norm, namely* $\|Vf\| = \|f\|$ *for any* $f \in \mathcal{H}$,

(iii) *If* $\mathcal{H} \neq \{0\}$ *then* $\|V\| = 1$,

(iv) VV^* *is the projection on* $\mathsf{Ran}(V)$,

(v) V *is invertible (in the sense of Definition 1.3.7)*,

(vi) *The adjoint* V^* *satisfies* $V^*f = V^{-1}f$ *if* $f \in \mathsf{Ran}(V)$, *and* $V^*g = 0$ *if* $g \perp \mathsf{Ran}(V)$.

b) An element $W \in \mathscr{B}(\mathcal{H})$ *is an isometry if and only if* $\|Wf\| = \|f\|$ *for all* $f \in \mathcal{H}$.

Exercise 1.4.7. *Provide a proof for the previous proposition (as well as the proof of the next proposition).*

More generally one defines *a partial isometry* as an element $W \in \mathscr{B}(\mathcal{H})$ such that

$$W^*W = P \tag{1.16}$$

with P an orthogonal projection. Again, unitary operators or isometries are special examples of partial isometries.

As before the following properties of partial isometries can be easily proved.

Proposition 1.4.8. *Let $W \in \mathscr{B}(\mathcal{H})$ be a partial isometry as defined in* (1.16)*. Then*

(i) one has $WP = W$ and $\langle Wf, Wg\rangle = \langle Pf, Pg\rangle$ for any $f, g \in \mathcal{H}$,

(ii) If $P \neq \mathbf{0}$ then $\|W\| = 1$,

(iii) WW^* is the projection on $\mathsf{Ran}(W)$.

For a partial isometry W one usually calls *initial set projection* the projection defined by W^*W and by *final set projection* the projection defined by WW^*.

Let us now introduce a last subset of bounded operators, namely the ideal of *compact operators*. For that purpose, consider first any family $\{g_j, h_j\}_{j=1}^N \subset \mathcal{H}$ and for any $f \in \mathcal{H}$ one sets

$$Af := \sum_{j=1}^{N} \langle g_j, f\rangle\, h_j. \tag{1.17}$$

Then $A \in \mathscr{B}(\mathcal{H})$, and $\mathsf{Ran}(A) \subset \mathsf{Vect}(h_1, \dots, h_N)$. Such an operator A is called *a finite rank operator*. In fact, any operator $B \in \mathscr{B}(\mathcal{H})$ with $\dim\big(\mathsf{Ran}(B)\big) < \infty$ is a finite rank operator.

Exercise 1.4.9. *For the operator A defined in* (1.17)*, give an upper estimate for $\|A\|$ and compute A^*.*

Definition 1.4.10. *An element $B \in \mathscr{B}(\mathcal{H})$ is* a compact operator *if there exists a family $\{A_n\}_{n\in\mathbb{N}}$ of finite rank operators such that $\lim_{n\to\infty} \|B - A_n\| = 0$. The set of all compact operators is denoted by $\mathscr{K}(\mathcal{H})$.*

The following proposition contains sone basic properties of $\mathscr{K}(\mathcal{H})$. Its proof can be obtained by playing with families of finite rank operators.

Proposition 1.4.11. *The following properties hold:*

(i) $B \in \mathscr{K}(\mathcal{H}) \Longleftrightarrow B^* \in \mathscr{K}(\mathcal{H})$,

(ii) $\mathscr{K}(\mathcal{H})$ is a $*$-algebra, complete for the norm $\|\cdot\|$,

(iii) If $B \in \mathscr{K}(\mathcal{H})$ and $A \in \mathscr{B}(\mathcal{H})$, then AB and BA belong to $\mathscr{K}(\mathcal{H})$.

As a consequence, $\mathscr{K}(\mathcal{H})$ is a C^*-algebra and an ideal of $\mathscr{B}(\mathcal{H})$. In fact, compact operators have the nice property of improving some convergences, as shown in the next statement.

Proposition 1.4.12. *Let $K \in \mathscr{K}(\mathcal{H})$).*

(i) If $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$ is a weakly convergent sequence with limit $f_\infty \in \mathcal{H}$, then the sequence $\{Kf_n\}_{n\in\mathbb{N}}$ strongly converges to Kf_∞,

(ii) If the sequence $\{B_n\}_{n\in\mathbb{N}} \subset \mathscr{B}(\mathcal{H})$ strongly converges to $B_\infty \in \mathscr{B}(\mathcal{H})$, then the sequences $\{B_nK\}_{n\in\mathbb{N}}$ and $\{KB_n^*\}_{n\in\mathbb{N}}$ converge in norm to $B_\infty K$ and KB_∞^*, respectively.

Proof. a) Let us first set $\varphi_n := f_n - f_\infty$ and observe that $\mathrm{w-}\lim_{n\to\infty}\varphi_n = 0$. By an application of the uniform boundedness principle, see Theorem 1.1.8, it follows that $\{\|\varphi_n\|\}_{n\in\mathbb{N}}$ is bounded, *i.e.* there exists $M > 0$ such that $\|\varphi_n\| \leq M$ for any $n \in \mathbb{N}$. Since K is compact, for any $\varepsilon > 0$ there exists a finite rank operator A of the form given in (1.17) such that $\|K - A\| \leq \frac{\varepsilon}{2M}$. Then one has

$$\|K\varphi_n\| \leq \|(K-A)\varphi_n\| + \|A\varphi_n\| \leq \frac{\varepsilon}{2} + \sum_{j=1}^{N} |\langle g_j, \varphi_n\rangle|\,\|h_j\|.$$

Since $\mathrm{w-}\lim_{n\to\infty}\varphi_n = 0$ there exists $n_0 \in \mathbb{N}$ such that $\langle g_j, \varphi_n\rangle|\,\|h_j\| \leq \frac{\varepsilon}{2N}$ for any $j \in \{1,\dots,N\}$ and all $n \geq n_0$. As a consequence, one infers that $\|K\varphi_n\| \leq \varepsilon$ for all $n \geq n_0$, or in other words $\mathrm{s-}\lim_{n\to\infty} K\varphi_n = 0$.

b) Let us set $C_n := B_n - B_\infty$ such that $\mathrm{s-}\lim_{n\to\infty} C_n = \mathbf{0}$. As before, there exists $M > 0$ such that $\|C_n\| \leq M$ for any $n \in \mathbb{N}$. For any $\varepsilon > 0$ consider a finite rank operator A of the form (1.17) such that $\|K - A\| \leq \frac{\varepsilon}{2M}$. Then observe that for any $f \in \mathcal{H}$

$$\begin{aligned}
\|C_nKf\| &\leq M\|(K-A)f\| + \|C_nAf\| \\
&\leq M\|K-A\|\,\|f\| + \sum_{j=1}^{N} |\langle g_j, f\rangle|\,\|C_nh_j\| \\
&\leq \Big\{M\|K-A\| \ + \sum_{j=1}^{N} \|g_j\|\,\|C_nh_j\|\Big\}\|f\|.
\end{aligned}$$

Since C_n strongly converges to $\mathbf{0}$ one can then choose $n_0 \in \mathbb{N}$ such that $\|g_j\|\,\|C_nh_j\| \leq \frac{\varepsilon}{2N}$ for any $j \in \{1,\dots N\}$ and all $n \geq n_0$. One then infers that $\|C_nK\| \leq \varepsilon$ for any $n \geq n_0$, which means that the sequence $\{C_nK\}_{n\in\mathbb{N}}$ uniformly converges to $\mathbf{0}$. The statement about $\{KB_n^*\}_{n\in\mathbb{N}}$ can be proved analogously by taking the equality $\|KB_n^* - KB_\infty^*\| = \|B_nK^* - B_\infty K^*\|$ into account and by remembering that K^* is compact as well. □

Exercise 1.4.13. *Check that a projection P is a compact operator if and only if $P\mathcal{H}$ is of finite dimension.*

Extension 1.4.14. *There are various subalgebras of $\mathscr{K}(\mathcal{H})$, for example the algebra of Hilbert-Schmidt operators, the algebra of trace class operators, and more generally the Schatten classes. Note that these algebras are not closed with respect to the norm $\|\cdot\|$ but with respect to some stronger norms $|||\cdot|||$. These algebras are ideals in $\mathscr{B}(\mathcal{H})$.*

1.5 Operator-valued maps

In analogy with Section 1.2 it is natural to consider function with values in $\mathscr{B}(\mathcal{H})$. More precisely, let J be an open interval on $\mathbb{R}$, and let us consider a map $F : J \to \mathscr{B}(\mathcal{H})$. The notion of continuity can be considered with several topologies on $\mathscr{B}(\mathcal{H})$, but as in Definition 1.3.5 we shall consider only three of them.

Definition 1.5.1. *The map F is* continuous in norm *on J if for all $t \in J$*

$$\lim_{\varepsilon \to 0} \big\|F(t+\varepsilon) - F(t)\big\| = 0.$$

The map F is strongly continuous *on J if for any $f \in \mathcal{H}$ and all $t \in J$*

$$\lim_{\varepsilon \to 0} \big\|F(t+\varepsilon)f - F(t)f\big\| = 0.$$

The map F is weakly continuous *on J if for any $f, g \in \mathcal{H}$ and all $t \in J$*

$$\lim_{\varepsilon \to 0} \big\langle g, \big(F(t+\varepsilon) - F(t)\big)f\big\rangle = 0.$$

One writes respectively $u - \lim_{\varepsilon \to 0} F(t+\varepsilon) = F(t)$, $s - \lim_{\varepsilon \to 0} F(t+\varepsilon) = F(t)$ and $w - \lim_{\varepsilon \to 0} F(t+\varepsilon) = F(t)$.

The same type of definition holds for the differentiability:

Definition 1.5.2. *The map F is* differentiable in norm *on J if there exists a map $F' : J \to \mathscr{B}(\mathcal{H})$ such that*

$$\lim_{\varepsilon \to 0} \Big\|\frac{1}{\varepsilon}\big(F(t+\varepsilon) - F(t)\big) - F'(t)\Big\| = 0.$$

The definitions for strongly differentiable *and* weakly differentiable *are similar.*

If J is an open interval of $\mathbb{R}$ and if $F : J \to \mathscr{B}(\mathcal{H})$, one defines $\int_J F(t)\,\mathrm{d}t$ as a Riemann integral (limit of finite sums over a partition of J) if this limiting procedure exists and is independent of the partitions of J. Note that these integrals can be defined in the weak topology, in the strong topology or in the norm topology (and in other topologies). For example, if $F : J \to \mathscr{B}(\mathcal{H})$ is strongly continuous and if $\int_J \|F(t)\|\mathrm{d}t < \infty$, then the integral $\int_J F(t)\,\mathrm{d}t$ exists in the strong topology.

Proposition 1.5.3. *Let J be an open interval of $\mathbb{R}$ and $F : J \to \mathscr{B}(\mathcal{H})$ such that $\int_J F(t)\,\mathrm{d}t$ exists (in an appropriate topology). Then,*

(i) For any $B \in \mathscr{B}(\mathcal{H})$ one has

$$B\int_J F(t)\,\mathrm{d}t = \int_J BF(t)\,\mathrm{d}t \quad \text{and} \quad \Big(\int_J F(t)\,\mathrm{d}t\Big)B = \int_J F(t)B\,\mathrm{d}t,$$

(ii) One has $\Big\|\int_J F(t)\,\mathrm{d}t\Big\| \le \int_J \|F(t)\|\,\mathrm{d}t$,

(iii) *If* $\mathscr{C} \subset \mathscr{B}(\mathcal{H})$ *is a subalgebra of* $\mathscr{B}(\mathcal{H})$*, closed with respect to a norm* $|||\cdot|||$*, and if the map* $F : J \to \mathscr{C}$ *is continuous with respect to this norm and satisfies* $\int_J |||F(t)||| \,\mathrm{d}t < \infty$*, then* $\int_J F(t)\,\mathrm{d}t$ *exists, belongs to* $\mathscr{C}$ *and satisfies*

$$\left|\left|\left| \int_J F(t)\,\mathrm{d}t \right|\right|\right| \leq \int_J |||F(t)|||\,\mathrm{d}t.$$

Note that the last statement is very useful, for example when $\mathscr{C} = \mathscr{K}(\mathcal{H})$ or for any Schatten class.