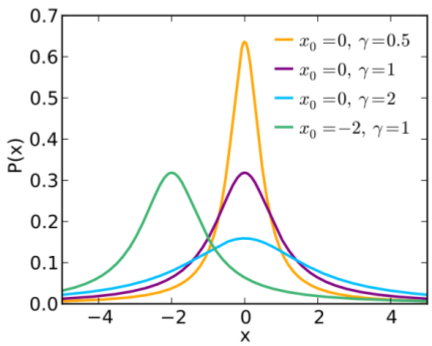
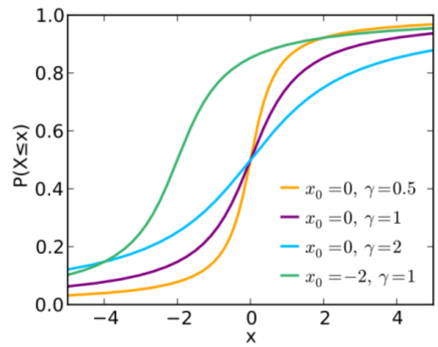
Cauchy Distribution





The Cauchy distribution, or the Lorentzian distribution, is a continuous probability distribution that is the ratio of two independent normally distributed random variables if the denominator distribution has mean zero. It is a "pathological" distribution, i.e. both its expected value and its variance are undefined.

$$pdf = \frac{1}{\pi\gamma[1 + (\frac{x - x_0}{\gamma})^2]}$$

where γ is the scale parameter.

The standard Cauchy distribution is also a special case of the Student's t-distribution with degrees of freedom $v = n - 1 = 1$, i.e. t(df = 1):

$$pdf = \frac{\Gamma\left(\frac{v+1}{2}\right)}{\Gamma\left(\frac{v}{2}\right)\sqrt{v\pi}}(1 + \frac{x^2}{v})^{-\frac{v+1}{2}} = \frac{1}{\pi(1 + x^2)}$$

where $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Some Properties:

• The pdf is symmetric about the line $x = x_0$ (which is also the median).

This is due to the pdf being an even function about $x = x_0$.

• The pdf has undefined mean.

Suppose the case of a standard Cauchy distribution, i.e., x_0 = 0, σ = 1. The mean of pdf is then

$$E(X) = \lim_{T_1 \to \infty} \lim_{T_2 \to \infty} \int_{-T_2}^{T_1} \frac{x}{\pi} \frac{1}{1+x^2} dx$$

which is an indefinite integral.

Suppose T_1 approaches ∞ at the rate of aT while T_2 at the rate of T, $then$

$$E(X)=\lim_{T\to\infty}\int_{-T}^{aT}\frac{x}{\pi}\frac{1}{1+x^2}dx = \lim_{T\to\infty}\frac{1}{2\pi}\ln\left(\frac{1+(aT)^2}{1+T^2}\right) = \frac{1}{2\pi}\lim_{T\to\infty}\ln\left(\frac{a+T^{-2}}{1+T^{-2}}\right) = \frac{1}{2\pi}\ln(a)$$

which is indefinite for different factors of a.

• The pdf has undefined variance.

The variance of a standard Cauchy distribution is

$$Var(X) = \int_{-\infty}^{\infty}\frac{1}{\pi}\frac{1}{1+x^2}(x-\mu)^2dx = \int_{-\infty}^{\infty}\frac{1}{\pi}\left(\frac{x^2}{1+x^2}-\frac{2\mu x}{1+x^2}+\frac{\mu^2}{1+x^2}\right)dx$$

where

$$\int_{-\infty}^{\infty}\frac{1}{\pi}\frac{x^2}{1+x^2}dx = \frac{1}{\pi}\int_{-\infty}^{\infty}\left(1-\frac{1}{1+x^2}\right)dx = \lim_{T\to\infty} x\,\Big|_{-T}^{aT} - \int_{-\infty}^{\infty}\frac{1}{1+x^2}dx$$

$$\int_{-\infty}^{\infty}\frac{2\mu x}{1+x^2}\frac{1}{\pi}dx = \frac{\mu}{\pi}\ln(1+x^2)\,\Big|_{-\infty}^{\infty}$$

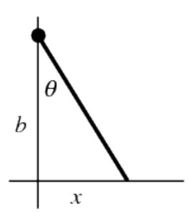
which approaches an indefinite value. Similarly, moments of the Cauchy distribution do not exist.

Applications:

• Density function for Cauchy processes.

• The Dirac delta function can be approximated by taking the limit of γ to 0..

• Distribution of horizontal distances at which the line segment tilted at a random angle cuts the x-axis:



• Modelling the points of impact of a fixed straight line of particles emitted from a point source.

• Special case of Student's t-distribution, where the sample size is 2.

• The distribution of the energy of an unstable state in quantum mechanics.