BINOMIAL DISTRIBUTION

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Def The binomial distribution with parameters n and p is the discrete probability distribution of the number of Successes in a sequence of n independent experiments, each asking a yes/no question, each with its own boolean-outcome: Success/yes/true (with probability p) or failure/no/false (with probability $q = 1-p$)

Applications: Used to model the number of successes in a sample size n drawn with replacement from a population of size N

⊛ Probability Mass Function

$$P(X=x \mid n,p) = P\left(\bigcup_{i=1}^{\binom{n}{x}} B_i\right)$$

with B_i: the i^{th} possible way to see x successes in n trials. $\forall i \neq j$: $B_i \cap B_j = \emptyset$.

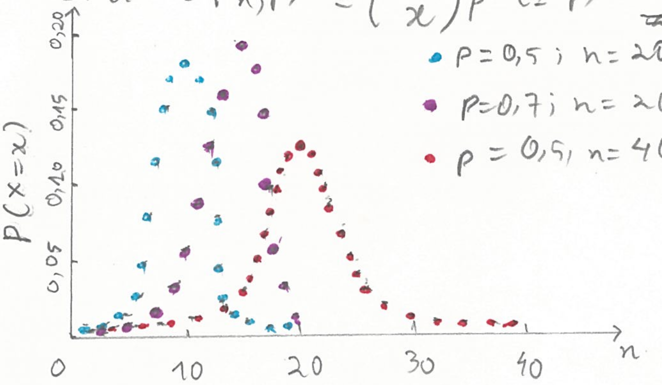
$$\rightarrow P(X=x \mid n,p) = \sum_{i=1}^{\binom{n}{x}} P(B_i) - \sum_{i \neq j; i,j=1}^{\binom{n}{x}} P(B_i \cap B_j) - P\left(\bigcap_{i=1}^{\binom{n}{x}} B_i\right)$$

$$= \sum_{i=1}^{\binom{n}{x}} P(B_i)$$

$$= \sum_{i=1}^{\binom{n}{x}} p^x (1-p)^{n-x}$$

$$= \binom{n}{x} p^x (1-p)^{n-x}$$

$$\boxed{P(X=x \mid n,p) = \binom{n}{x} p^x (1-p)^{n-x}}$$



⊛ Variance

$$E(X(X-1)) = \sum_{x=0}^{n} x(x-1)\binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=2}^{n} \frac{n!}{(x-2)!(n-x)!} p^x (1-p)^{n-x}$$

$$= n(n-1)p^2 \sum_{y=0}^{m} \frac{m!}{y!(m-y)!} p^y (1-p)^{m-y}$$

with $y = x-2$; $m = n-2$

$$\rightarrow E(X(X-1)) = n(n-1)p^2 \sum_{y=0}^{m} \binom{m}{y} p^y (1-p)^{m-y}$$

$$= n(n-1)p^2 (1-p+p)^m$$

$$= n(n-1)p^2$$

$$Var(X) = E(X^2) - E(X)^2$$

$$= E(X(X-1)) + E(X) - E(X)^2$$

$$= n(n-1)p^2 + np - (np)^2$$

$$= np(1-p)$$

$$\boxed{Var(X) = np(1-p)}$$

⊛ Approximation:

For small p, the binomial distribution can be approximated by Poisson distribution

(more exactly, in the limit $N \to \infty$ and $p \to 0$, $N \cdot p$ = constant)

⊛ Mean value

$$E(X) = \sum_{x=0}^{n} x\binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=1}^{n} \frac{n!}{(x-1)!(n-x)!} p^x (1-p)^{n-x}$$

Set $y = x-1$; $m = n-1$

$$\rightarrow E(X) = \sum_{y=0}^{m} \frac{(m+1)!}{y!(m-y)!} p^{y+1} (1-p)^{m-y}$$

$$= (m+1)p \sum_{y=0}^{m} \frac{(m)!}{y!(m-y)!} p^y (1-p)^{m-y}$$

$$= np \sum_{y=0}^{m} \binom{m}{y} p^y (1-p)^{m-y}$$

$$= np \, (p + (1-p))^m$$

$$= np$$

$$\boxed{E(X) = np}$$

⊛ Moment Generating Function

$$M_X(t) = E(e^{tX})$$

$$= \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} (pe^t)^x (1-p)^{n-x} \binom{n}{x}$$

$$= [pe^t + (1-p)]^n$$

$$\boxed{M_X(t) = [pe^t + (1-p)]^n}$$

⊛ Related to Binomial Theorem:

$x, y \in \mathbb{R}$; $n \in \mathbb{Z}^+$

$$(x+y) = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

Proof: $(x+y)^n = (x+y)(x+y) \cdots (x+y)$

From each factor $(x+y)$, choose either x and y (n choices). For each $i = 0, 1, \ldots, n$, number of terms that x appears i times is $\binom{n}{i}$, thus this term is of the form $\binom{n}{i} x^i y^{n-i}$

As we take the sum of those terms

$$\rightarrow (x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

As we choose $x = p$; $y = 1-p$

$$\rightarrow 1 = (p + (1-p))^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i} = \sum_{x=0}^{n} P(X=x \mid n,p)$$