# Likelihood ratio test

## Appendix 7

<u>Aim</u> : testing hypothesis about mean of a normal random sample with unknown variance.

unknown

Let $X = (X_1, \dots, X_N)$ with $X_j \sim n(\mu, \sigma^2)$ be the random sample, and let $H_0 : \mu = \mu_0$ for a given $\mu_0 \in \mathbb{R}$. On the other hand, $\sigma^2 > 0$ is arbitrary. Set $\Theta := (\mu, \sigma^2)$.

Recall that $L(\Theta | \underline{x}) = \left(\frac{1}{2\pi\sigma^2}\right)^{N/2} \exp\left(-\sum_{j=1}^{N}(x_j - \mu)^2 / 2\sigma^2\right)$.

The maximum of $\Theta \mapsto L(\Theta | \underline{x})$ is realized for $\mu = \bar{x}$ and $\sigma^2 = \frac{1}{N}\sum_{j=1}^{N}(x_j - \bar{x})^2$, see Example 7.2.11 in [CB]. By the same argument one gets that if $\mu = \mu_0$, the maximum of $\sigma^2 \mapsto L(\mu_0, \sigma^2 | \underline{x})$ is realized for $\sigma^2 = \frac{1}{N}\sum_j (x_j - \mu_0)^2$. These results are obtained by looking at the critical points of $\Theta \mapsto L(\Theta | \underline{x})$.

Then, set $\sigma_0^2 := \frac{1}{N}\sum_j (x_j - \mu_0)^2$ and $\sigma^2 := \frac{1}{N}\sum_j (x_j - \bar{x})^2$.

$$\lambda(\underline{x}) := \frac{\sup_{\sigma^2>0} L(\mu_0, \sigma^2 \mid \underline{x})}{\sup_{\mu, \sigma^2} L(\mu, \sigma^2 \mid \underline{x})}$$

$$= \frac{(2\pi\sigma_0^2)^{-N/2} \exp\left(-\sum_j (x_j-\mu_0)^2 / 2\sigma_0^2\right)}{(2\pi\sigma^2)^{-N/2} \exp\left(-\sum_j (x_j-\bar{x})^2/2\sigma^2\right)}$$

$$= \left(\frac{\sigma^2}{\sigma_0^2}\right)^{N/2} \frac{e^{-N/2}}{e^{-N/2}}$$

$$= \left(\frac{\sigma^2}{\sigma_0^2}\right)^{N/2}$$

$$= \left(\frac{\sum_j (x_j-\bar{x})^2}{\sum_j (x_j-\mu_0)^2}\right)^{N/2}.$$

not known but fixed

Thus, one would reject $H_0$ if $\lambda(\underline{x}) \leq c$ for a given $c \in (0,1)$. But let us go one more step. Observe that $\sum_j (x_j-\mu_0)^2 = \sum_j (x_j-\bar{x})^2 + N(\bar{x}-\mu_0)^2$.

Then $\lambda(\underline{x}) \leq c \iff \left(\frac{\sum_j (x_j-\bar{x})^2}{\sum_j (x_j-\mu_0)^2}\right)^{N/2} \leq c$

$$\iff \frac{\sum_j (x_j-\bar{x})^2}{\sum_j (x_j-\bar{x})^2 + N(\bar{x}-\mu_0)^2} \leq c^{2/N}$$

$$\iff \frac{1}{1 + \frac{N(\bar{x}-\mu_0)^2}{\sum_j (x_j-\bar{x})^2}} \leq c^{2/N}$$

$$\Longleftrightarrow \quad \frac{N(\bar{x}-\mu_0)^2}{\sum_j (x_j-\bar{x})^2} \geq \frac{1}{c^{2/N}} - 1$$

sample mean

$$\Longleftrightarrow \quad \frac{(\sqrt{N}(\bar{x}-\mu_0))^2}{\underbrace{\frac{1}{N-1}\sum_j (x_j-\bar{x})^2}_{\text{sample variance } S^2}} \geq (N-1)(c^{-2/N}-1)$$

$$\Longleftrightarrow \quad \frac{|\bar{x}-\mu_0|}{S/\sqrt{N}} \geq \underbrace{\left((N-1)(c^{-2/N}-1)\right)^{1/2}}_{=: c'}$$

follow a student t-distribution with parameter $N-1$.

Thus, $H_0$ would be rejected if $\frac{|\bar{x}-\mu_0|}{S/\sqrt{N}} \geq c'$.

Conclusion: The likelihood ratio test is equivalent to a t-test (in this setting).

Some useful exercises:

Example 7.2.11 + 7.2.12
Example 8.2.3
Thm 8.2.4 about sufficient statistics
Exercise 8.3
Example 8.2.9