Inequalities

Appendix 3

see [CB], sec. 4.7

Lemma (Markov inequality) If $X \geq 0$, then
$$P(X > t) \leq \frac{E(X)}{t} \quad \forall t > 0.$$

$\Rightarrow$ if $\mu := E(X)$ and $\sigma := Var(X)$, $P\left(\frac{(X-\mu)^2}{\sigma^2} \geq t^2\right) \leq \frac{1}{t^2}$.
(Chebychev inequality)

Lemma (key of many inequalities)
If $a, b \geq 0$ and $p, q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$

then $\frac{1}{p} a^p + \frac{1}{q} b^q \geq ab$

this is a generalization of $\frac{1}{2}a^2 + \frac{1}{2}b^2 \geq ab$
$\Leftrightarrow$ $2ab \leq a^2 + b^2$ (obtained from $(a-b)^2 \geq 0$).

Thm (Hölder's inequality)
Let X, Y be 2 random variables, and $p, q > 1$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$|E(XY)| \leq E(|XY|) \leq (E(|X|^p))^{1/p} (E(|Y|^q))^{1/q}. \quad (*)$$

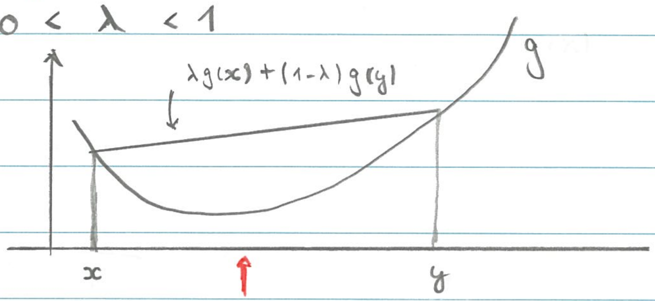
When $p = q = 2$, one gets the Cauchy-Schwarz inequality:

$$|E(XY)| \leq E(|XY|) \leq (E(|X|^2))^{1/2} (E(|Y|^2))^{1/2},$$

From (*) one can also deduce the Lyapounov's inequality:
$$(E(|X|^r))^{1/r} \leq (E(|X|^s))^{1/s}$$
for $1 < r < s < \infty$.

<u>Thm</u> (Minkowski's inequality)
Let X, Y be random variables, and $1 \leq p < \infty$.
Then
$$\left(E(|X+Y|^p) \right)^{1/p} \leq \left(E(|X|^p) \right)^{1/p} + \left(E(|Y|^p) \right)^{1/p}.$$

Recall that a function $g : \mathbb{R} \to \mathbb{R}$ is <u>convex</u>
if $g(\underbrace{\lambda x + (1-\lambda) y}_{*}) \leq \lambda g(x) + (1-\lambda) g(y) \quad \forall x, y \in \mathbb{R}$
and $0 < \lambda < 1$



* any point between x and y

<u>Thm</u> (Jensen's inequality)
Let X be a random variable, and $g : \mathbb{R} \to \mathbb{R}$, Borel measurable and convex. Then
$$E(g(X)) \geq g(E(X)).$$

One directly deduces : $E(X^2) \geq (E(X))^2$
and $E(1/X) \geq 1/E(X)$.

If $a_1, \dots, a_N \in \mathbb{R}_+$, one sets
arithmetic mean : $a_A := \frac{1}{N}(a_1 + a_2 + \dots + a_N)$
geometric mean : $a_G := (a_1 a_2 \dots a_N)^{1/N}$
harmonic mean : $a_H = n \left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_N} \right)^{-1}$
and one has $a_H \leq a_G \leq a_A$.

Convergence

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables. There exist several types of convergence of this sequence to a random variable X_∞.

<u>Def</u>: $(X_n)_{n\in\mathbb{N}}$ converges to X_∞ <u>in probability</u> if $\forall \varepsilon > 0$, $P(|X_n - X_\infty| \geq \varepsilon) \xrightarrow[n\to\infty]{} 0$.

<u>Def</u>: $(X_n)_{n\in\mathbb{N}}$ converges to X_∞ <u>almost surely</u> if $P(\lim X_n = X_\infty) = 1$

$$\Leftrightarrow P(\{s \in S \mid \lim_{n\to\infty} X_n(s) = X_\infty(s)\}) = 1.$$

<u>Example</u>: Let $(X_i)_{i\in\mathbb{N}}$ be iid with $E(X_i) = \mu$ and $Var(X_i) = \sigma^2$. Set $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$.

Weak law of large number: $\overline{X}_n \to \mu$ in probability

Strong law of large number: $\overline{X}_n \to \mu$ almost surely

discrete random variable taking only the value μ.

<u>Remarks</u>:

1) If $X_n \to X_\infty$ in probability and $g: \mathbb{R} \to \mathbb{R}$ is continuous, then $g(X_n) \to g(X_\infty)$ in probability.

2) Almost sure convergence $\Rightarrow$ convergence in probability.

<u>Def</u> : $(X_n)_{n\in\mathbb{N}}$ converges in distribution to X_∞ if $\lim_{n\to\infty} F_{X_n}(x) = F_{X_\infty}(x)$ at all $x \in \mathbb{R}$ with F_{X_∞} continuous at x.

cdf = cumulative distribution function

<u>Remark</u>: Convergence in probability $\Rightarrow$ convergence in distribution

$\Leftarrow$ if X_∞ is a constant random variable.

<u>Central limit theorem</u> :

Let $(X_i)_{i\in\mathbb{N}}$ be iid with $E(X_i) = \mu$ and $Var(X_i) = \sigma^2 \in (0,\infty)$. Set $\overline{X}_n := \frac{1}{n}\sum_{i=1}^{n} X_i$, and $Z_n := \sqrt{n}\,\frac{\overline{X}_n - \mu}{\sigma}$. Then

$$F_{Z_n}(x) \xrightarrow[n\to\infty]{} \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dy ,$$

that is Z_n converges in distribution to the random variable having the standard normal distribution.

<u>Lemma</u> (Slutsky's theorem)

Let $X_n \to X_\infty$ in distribution, and $Y_n \to a$ in probability, with a a constant r.v. Then

1) $X_n Y_n \to a X_\infty$ in distribution

2) $X_n + Y_n \to X_\infty + a$ in distribution.

A few exercises

- Study and summerized the hierarchical models, as presented in Section 4.4 and in its examples.
- Study Examples 4.2.2 and 4.2.4 about conditional pdf or pmf.
- Study the proof of thm 4.5.7 about the correlation ρ_{XY} of X and Y.
- Study the proof of Thm 5.2.6 about sample mean, variance and standard deviation.
- Study the proof of thm 5.3.1 about a family of normal distribution.
- Study the student's t distribution as presented in Section 5.3.2.