Cong Bui
Song Ha

On $1-\alpha$ confidence interval

<u>Question</u>: Find a and b such that: $\left(\frac{1}{a}\right)^N - \left(\frac{1}{b}\right)^N = 1-\alpha$ and $(b-a)$ is minimum among $1 \leq a < b$, $\alpha \in [0,1]$

<u>Solution</u>:

+) Let $x := a^N$, $y := b^N$

$$\Rightarrow \begin{cases} 1 \leq x < y \\ \frac{1}{x} - \frac{1}{y} = 1-\alpha \Rightarrow x = \dfrac{y}{1+(1-\alpha)y} \end{cases}$$

+) Since $x \geq 1 \Rightarrow \dfrac{y}{1+(1-\alpha)y} \geq 1$

$$\Rightarrow y \geq 1+(1-\alpha)y$$

$$\Rightarrow y \geq \frac{1}{\alpha} \qquad (1)$$

+) One has: $b-a = \sqrt[N]{y} - \sqrt[N]{\dfrac{y}{1+(1-\alpha)y}}$

Let RHS $:= f(y)$

$$\Rightarrow f'(y) = \frac{1}{N\sqrt[N]{y^{N-1}}}\left(1 - \frac{1}{\sqrt[N]{[1+(1-\alpha)y]^{N+1}}}\right) > 0, \text{ since } (1-\alpha)y > 0 \text{ and } y > 0$$

$\Rightarrow f(y)$ is increasing $\qquad (2)$

+) From (1) and (2) $\Rightarrow f(y) \geq f\left(\frac{1}{\alpha}\right) = \sqrt[N]{\frac{1}{\alpha}} - 1$

$$\Rightarrow b-a \geq \sqrt[N]{\frac{1}{\alpha}} - 1$$

"=" holds $\Leftrightarrow \begin{cases} a = 1 \\ b = \sqrt[N]{\frac{1}{\alpha}} \end{cases}$. $\square$