Lie groups and Lie algebras are useful in QM and in particle physics.

Lemma: Let $T: G \mapsto \mathcal{B}(\mathcal{H})$ with G a compact group, s.t. this map is weakly continuous.

Weakly continuous :⇔ $G \ni a \mapsto \underbrace{\langle f, T(a)g \rangle}_{\text{inner product on Hilbert space } \mathcal{H}} \in \mathbb{C}$ is continuous $\forall f, g \in \mathcal{H}$)

Then $\exists \mathbf{T} \in \mathcal{B}(\mathcal{H}): \langle f, \mathbf{T}g \rangle = \int_G \langle f, T(a)g \rangle \, da \ \forall f, g \in \mathcal{H}$.

It means that $\mathbf{T} = \int_G T(a) \, da$ different from the integral introduced in the last lecture? {on $C(G)$ / on representation

Idea of the proof:

Choose $\{f_j\}_{j \in \mathbb{N}}$ basis of $\mathcal{H}$ and compute $\alpha_{jk} := \int \langle f_j, T(a) f_k \rangle \, da$, and we set $\langle f_j, \mathbf{T} f_k \rangle := \alpha_{jk}$. Then we should check $\mathbf{T}$ is a bounded operator.

To give a meaning of $\int_G U(a) T U(a)^{-1} da$

For Lie group G we shall consider unitary representations (it means $\forall a \in G: U(a) \in \mathcal{B}(\mathcal{H})$ is unitary)

which are strongly continuous, it means $G \ni a \mapsto U(a) f \in \mathcal{H}$ is continuous

:⇔ $\lim_{b \to a} \| U(b) f - U(a) f \| = 0 \ \forall f \in \mathcal{H}$

When using ε-δ, δ can be dependent on f.

(If continuous on norm, δ is independent on f)

can be defined by the local charts

Corollory: If $U: G \mapsto \mathcal{U}(\mathcal{H})$ (set of unitary operators on $\mathcal{H}$) is strongly continuous and G a compact Lie group,

Proof as exercise

Then for any $T \in \mathcal{B}(\mathcal{H})$: the map $G \ni a \mapsto \overset{\text{SC}}{U(a)} \overset{\text{any}}{T} U(a)^{-1}$ is strongly continuous (⇒ weakly continuous) and

then $M_G(T) := \int U(a) T U(a)^{-1} da \in \mathcal{B}(\mathcal{H})$

$M_G(T)$ satisfies $U(b) M_G(T) U(b)^{-1} = M_G(T) \ \forall b \in G$

A deep thm (Peter-Weyl theorem (Part II))

Let $U: G \mapsto \mathcal{U}(\mathcal{H})$ be a strongly continuous unitary representation of a compact Lie group G.

Then $\mathcal{H} = \bigoplus_{n \in \mathbb{N}} \mathcal{H}_n$ with $\dim \mathcal{H}_n < \infty$ ¿Est-ce possible qu'il y a infini d'$\mathcal{H}_n$?

and $U|_{\mathcal{H}_n}$ is an irreducible representation of G.

⇒ All irreducible representations of a compact Lie group are finite dimensional.

We denote by $\eta_k = [(U_k, \mathcal{H}_k)]$ the equivalence class containing the irreducible representation $(U_k, \mathcal{H}_k)$. ~~We set~~

Prop. Let $(U, \mathcal{H})$ be any finite dim rep of G. proof as exercise

Recall the character $\chi(a) := tr(U(a))$

1) $\mathcal{H} = \oplus \nu_k \mathcal{H}_k$ with $\nu_k = \int_G \overline{\chi(b)} \chi_k(b)\, db$

2) $(U, \mathcal{H})$ is irreducible iff $\int |\chi(b)|^2 db = 1$

3) $\int \overline{\chi_k(b)} \chi_\ell(b)\, db = \delta_{k\ell}$

4) $\int \overline{U(b)^\ell_{st}}\, U(b)^k_{pq}\, db = \delta_{\ell k} \delta_{sp} \delta_{tq}$

These relations can still be interpreted as orthogonal relations in $L^2(G)$

$L^2(G) := \{f : G \mapsto \mathbb{C} \mid \int |f(b)|^2 db < \infty\}$ (L^2-integrable on G)

Note that $\dim(L^2(G)) = \infty$

Now we pay attention to the props around the identity of a Lie group. (which leads to Lie-Algebra)

Recall that G is CONNECTED if any $a, b \in G$ can be connected by a continuous path, it means $\exists f : [0,1] \mapsto G$ continuous with $f(0) = a$, $f(1) = b$.

G is SIMPLY CONNECTED if any closed curve can be deformed to a point in G.

In a Lie group, the IDENTITY COMPONENT $G_0 \equiv G_e$ is the set of all elements of G which are connected to e.

Prop. 1) $G_0 \triangleleft G$ proof as exercise

2) If $(U, \mathcal{H})$ is a representation or a projective representation of G, Then G_0 is always represented by unitary operators.

Proof of 2)

$\forall a \in G_e : a = a_1^2 a_2^2 \cdots a_N^2$ for a finite family of elements in G_0. will be shown later

Then we observe $U(a_1^2) = w(a_1, a_1) U(a_1) U(a_1)$ with

$U(a_1)$ is $\begin{cases} \text{unitary} & :\Leftrightarrow \langle Uf, Ug\rangle = \langle f, g\rangle \Rightarrow U(a_1)U(a_1) \text{ is unitary} \\ \text{or} & \\ \text{anti-unitary} & :\Leftrightarrow \langle Uf, Ug\rangle = \langle g, f\rangle \Rightarrow U(a_1)U(a_1) \text{ is unitary} \end{cases}$

$\Rightarrow U(a)$ is unitary. $\square$

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III.2 Linear (or matrix) Lie groups and Lie algebra

⚠ Most (if not all) statements are true for general Lie groups, but the statements are more complicated.

Def. A LINEAR (or MATRIX) LIE GROUP is a Lie subgroup of $GL(n,\mathbb{C})$ or $GL(n,\mathbb{R})$

⚠ Even in $GL(n,\mathbb{C})$ we consider some parametrization with real coefficients. (up to $2n^2$ parameters)

· Topology on $GL(n,\mathbb{C})$ is induced by the distance ($\|\cdot\|_2$)

$$d : GL(n,\mathbb{C})\times GL(n,\mathbb{C}),\ d(A,B) := \Big(\sum_{j,k=1}^{n}|a_{jk}-b_{jk}|^2\Big)^{\frac{1}{2}} \text{ with } A=(a_{jk}),\ B=(b_{jk})$$

Let G be a Lie subgroup of $GL(n,\mathbb{C})$ and let (U_0,φ_0) be a chart near $e=\mathbb{1}\in G$.

Suppose (for simplicity) $\varphi_0(e)=\mathbf{0}\in\mathbb{R}^m$. Let us set

$$Y_j := \lim_{\varepsilon\to 0}\frac{\varphi_0^{-1}(\varepsilon E_j)-\mathbf{1}}{\varepsilon}\in M_n(\mathbb{C}) \qquad \text{for } j=1,\cdots,m$$

Here the linear assumption plays a role:

$$(Y_j)_{\ell m} = [\partial_j \underbrace{(\varphi_0^{-1})_{\ell m}}_{:\mathbb{R}^m\mapsto\mathbb{C}}](0)$$

Facts:
proof as exercise

1) The matrices $Y_1,\cdots,Y_m$ are linearly independent (over $\mathbb{R}$) $\quad \alpha Y_j+\beta Y_k=0$ && $\alpha,\beta\in\mathbb{R}$ $\Leftrightarrow \alpha=\beta=0$

(1 and i are linearly independent over $\mathbb{R}$ but not over $\mathbb{C}$)

but each Y_j can be made of complex numbers.

2) If $(-1,1)\ni t\mapsto X(t)\in\mathbb{R}^m$ with $X(0)=0$ is smooth

Then $(-1,1)\ni t\mapsto \varphi_0^{-1}\circ X(t)\in G$ is a smooth map and

$$\varphi_0^{-1}\circ X(0)=e \text{ and } \Big[\frac{d}{dt}\varphi_0^{-1}\circ X(t)\Big](0) = \sum_{j=1}^{m} Y_j X_j'(0)$$

It means that the derivative at 0 of any smooth curve in G passing to e at 0 is a linear combination (over $\mathbb{R}$) of $\{Y_j\}_{j=1}^m$.

The vector space generated by $\{Y_j\}$ is called the TANGENT SPACE at e.

Note: Tangent space is defined in any Lie group.