III. Lie groups

III.1: Main Definitions and Properties

Examples: $O(3)$, $SU(n)$, $SO(25)$ are Lie groups.

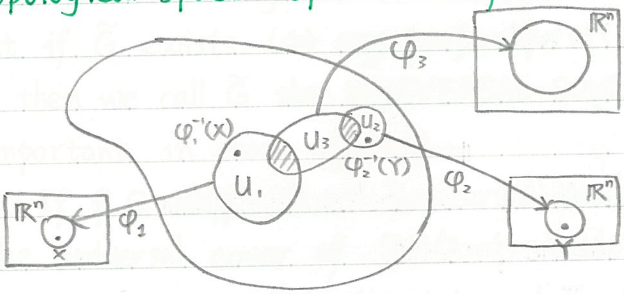
Roughly, a lie group is a group together with a differiential structure compatible with the group operation and with taking the inverse.

Def. G is a SMOOTH MANIFOLD if G is a second-countable Hausdalf topologica[l] space with local bijective and bi-continuous maps from the manifold to $\mathbb{R}$ s.t. $\varphi_j^{-1}\circ\varphi_i$ and $\varphi_i^{-1}\circ\varphi_j$ are C^∞ where def.

↳: M and M^{-1} are continuous

Hausdalf: 2 points always have disjoint neighborhood ⓧ ⓧ

Topological space: Open sets defined



$\varphi_1^{-1}\circ\varphi_3$ defined on a subset of $\varphi_1(U_1)$

A manifold is not necessarily a group.

Also (U_j, φ_j) is called a LOCAL CHART,

$\{(U_j,\varphi_j)\}$ with the compatibility $(\varphi_j^{-1}\circ\varphi_i,\ \varphi_i^{-1}\circ\varphi_j \in C^\infty)$ conditions is called an ATLAS, and we have

$$G = \bigcup_j U_j.$$

⇒ Locally the manifold can be parameterized by n real parameters.

When G is also a group, and the map

$$G\times G \ni (a,b) \mapsto ab \in G \quad \text{and} \quad G \ni a \mapsto a^{-1} \in G \text{ are smooth,}$$

It means $\forall (U_j, \varphi_j)$ for $j\in\{1,2,3\}$

1) $\Omega_{123} := \{(X,Y)\in\varphi_1(U_1)\times\varphi_2(U_2) \mid \varphi_1^{-1}(X)\varphi_2^{-1}(Y)\in U_3\} \subset \mathbb{R}^{2n}$ and then
$\Omega_{123} \ni (X,Y) \mapsto \varphi_3(\varphi_1^{-1}(X)\varphi_2^{-1}(Y)) \in \mathbb{R}^n$ is smooth;

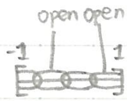
2) $\Omega_{12} := \{X \in \varphi_1(U_1) \mid (\varphi_1^{-1}(X))^{-1} \in U_2\} \subset \mathbb{R}^n$ and then
$\Omega_{12} \ni X \mapsto \varphi_2((\varphi_1^{-1}(X))^{-1}) \in \mathbb{R}^n$ is smooth.

Then G is a LIE GROUP.

Examples

$(\mathbb{R},+)$, $(\mathbb{R}_+,\cdot)$, $(\mathbb{T},\cdot)$ are Lie groups, with $\mathbb{T} = \{z\in\mathbb{C} \mid |z|=1\}$

Def. G is a COMPACT LIE GROUP if it is a group and
every cover of G by open sets admits a finite subcover.
For subsets of $\mathbb{R}^n$, compactness means bounded and open.



$\Rightarrow$ $(\mathbb{T}, \cdot)$ is a compact Lie group but $(\mathbb{R}, +)$ and $(\mathbb{R}_+, \cdot)$ are not.

A fundamental property of compact Lie group G:

Prop. $C(G) := \{f : G \mapsto \mathbb{C} \mid f \text{ is continuous}\}$

$\exists I : C(G) \mapsto \mathbb{C}$ s.t.

1) $I(f_1 + \alpha f_2) = I(f_1) + \alpha I(f_2)$ $\forall f_j \in C(G), \alpha \in G$ Linearity
2) $f \geqslant 0 \Rightarrow I(f) \geqslant 0$ and $f = 1 \Rightarrow I(f) = 1$ Positivity & Normalization
3) $I(f(a\cdot)) = I(f) = I(f(\cdot a))$ Invariance under Left & Right Multiplication
4) $I(f(\cdot^{-1})) = I(f)$ Invariance under Taking the Inverse

$\Rightarrow$ This map I corresponds to "integration" and is often denoted by
$\int_G f(a)\, da$ (called the Haar measure)

The Haar measure can be explicitely constructed locally
if G is not compact (but ~~not~~ locally compact), and
such a measure also exists but with less properties
($f = 1 \not\Rightarrow I(f) = 1$;
(3)(4) are not true and it is divided to left Haar & right Haar measures).

A Haar measure is very convenient because

$$\frac{1}{g} \sum_{a \in G} U(a) T U(a)^{-1} \text{ (if G is finite)} \rightsquigarrow \int_G U(a) T U(a)^{-1} da$$

↳ $\in \mathcal{B}(\mathcal{H})$ ↳ $\in L(V)$ linear maps on a vector space