Consider U a unitary operator ~~on~~ in $\mathcal{H}$ ($\Leftrightarrow U \in B(\mathcal{H}), U^* = U^{-1}$)
And set $S_U : \hat{\mathcal{H}} \mapsto \hat{\mathcal{H}}$, $S_U \hat{\varphi} = \widehat{U\psi}$ with $\psi \in \hat{\varphi}$
Then S_U is a symmetry. Indeed
Take $f = \alpha U \psi$ with $1 = \|f\| = |\alpha| \, \|U\psi\| = |\alpha| \, \|\psi\|$
$g = \beta U \varphi$ with $1 = \|g\| = |\beta| \, \|\varphi\|$
Then $T_r(P_{S\hat{\psi}} P_{S\hat{\varphi}}) = |\langle f, g\rangle|^2 = |\langle \alpha U \psi, \beta U \varphi\rangle| = |\langle \alpha\psi, \beta\varphi\rangle| = T_r(P_{\hat{\psi}} P_{\hat{\varphi}})$
The same holds if U is anti-unitary
(which means $U(f + \alpha g) = Uf + \bar{\alpha} Ug$, $\langle Uf, Ug\rangle = \overline{\langle f, g\rangle}$)

Thm. (Wigner's Thm)
Let $S : \hat{\mathcal{H}} \mapsto \hat{\mathcal{H}}$ be a symmetry.
Then $\exists U : \mathcal{H} \mapsto \mathcal{H}$ ~~with~~ which is either unitary or anti-unitary s.t. $S = S_U$.
U is unique modulo (up to) $\alpha \in \mathbb{C}$ with $|\alpha| = 1$,
since U and αU define the same symmetry.

Now consider a group of symmetries, which means a
homomorphism $S : G \mapsto \{\text{symmetries on } \mathcal{H}\}$
(each $S(a) : \mathcal{H} \mapsto \mathcal{H}$)
s.t. $S(a) S(b) = S(ab)$ $\forall a, b \in G$; $S(e) = 1$

By Wigner's Thm, $\forall a \in G \, \exists U(a)$ unitary or anti-unitary s.t. $S(a) = S_{U(a)}$.
Suppose all $U(a)$ are unitary.
Natual question: $U(a) U(b) = U(ab)$? No in general (problem of phase)
↳ Projective projections

Indeed if we fix $U(\cdot)$ we usually have
$U(a) U(b) = \omega(a,b) U(ab)$ with $\omega(a,b) \in \mathbb{C}$, $|\omega(a,b)| = 1$
and if we set $U'(a) = \rho(a) U(a)$ with $\rho : G \mapsto \mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$
then $U'(a) U'(b) = \rho(a) U(a) \rho(b) U(b) = \rho(a)\rho(b)\omega(a,b) U(ab)$
$= \frac{\rho(a)\rho(b)}{\rho(ab)} \omega(a,b) U'(ab)$
$= \omega'(a,b) U'(ab)$
$\Rightarrow \omega'(a,b) = \frac{\rho(a)\rho(b)}{\rho(ab)} \omega(a,b)$

Def. Let V be a complex vector space and let G be a group.

A PROJECTIVE REPRESENTATION ~~is~~ of G in V is a map

$U : G \mapsto GL(V)$ (→ set of invertible operators in V) s.t.

$U(a)U(b) = \omega(a,b)\, U(ab)$ with $\omega(a,b) \in \mathbb{C}^*$ and

$U(e) = 1$. called 2-cocycle (Ma) or Schur (Ph) multiplier.

Def. Two 2-cocycles $\omega, \omega' : G \times G \mapsto \mathbb{C}^*$ are EQUIVALENT if

$\exists \rho : G \mapsto \mathbb{C}^* : \omega'(a,b) = \frac{\rho(a)\rho(b)}{\rho(ab)}\, \omega(a,b) \ \forall a,b \in G$

ω is TRIVIAL if it is equivalent to 1, which means

$\exists \rho : G \mapsto \mathbb{C}^* : \omega(a,b) = \frac{\rho(a)\rho(b)}{\rho(ab)}$

Remarks

- Projective representations are essential in QM.
- A linear representation ~~if~~ is a special case of a proj. rep.
- If ω is trivial, then by setting $U'(a) = \rho(a)^{-1} U(a)$ we get a linear representation. Indeed,

$$U'(a)U'(b) = \underbrace{\rho(ab)\rho(a)^{-1}\rho(b)^{-1}\omega(ab)}_{=1}\rho(ab)^{-1}U(ab) = U'(ab)$$

- If $(U, \mathcal{H}), (U', \mathcal{H})$ are proj. rep. and if $U'(a) = \rho(a)U(a)$ $\exists \rho : G \mapsto \mathbb{C}^*$ then the 2 proj. rep. are EQUIVILENT.

Question: Can one always trivialize a 2-cocycle?

($\Leftrightarrow$ Do we always have $\exists \rho : G \mapsto \mathbb{C}^* : \omega(a,b) = \frac{\rho(a)\rho(b)}{\rho(ab)}$?) No

For a given group, the answer is in the study of GROUP COHOMOLOGY.

What is often useful is to consider a larger group and its lin. rep.

Example: Consider G and a second group $\tilde{G}$ with a normal subgroup $\tilde{G}_0$ s.t.

$\tilde{G}/\tilde{G}_0 \simeq G$ (Let's denote by $\phi : \tilde{G}/\tilde{G}_0 \mapsto G$ the isomorphism)

Let $(\tilde{U}, \mathcal{H})$ be a linear rep. of $\tilde{G}$ s.t.

$\forall a \in \tilde{G}_0 : \tilde{U}(a) = \sigma(a)\mathbb{1}$ $\exists \sigma(a) \in \mathbb{C}^*$ ($\Leftrightarrow \tilde{U}(\tilde{G}_0) \subset \mathbb{C}^*\mathbb{1}$)

$\forall a \in G$ let's choose $\tilde{a} \in \tilde{G}$ with $\phi([\tilde{a}]_{G_0}) = a$ and ~~U : G ↦ B(H) with U(a) :=~~

$U : G \mapsto \mathcal{B}(\mathcal{H})$ with $U(a) := \tilde{U}(\tilde{a})$.

Then U defines a projective rep. of G.

No.

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Indeed let $a, b \in G$ and set $d := ab$. Consider $\tilde{a}, \tilde{b}, \tilde{d}$ in $\tilde{G}$. Then

$$\phi([\tilde{d}]_{\tilde{G}_0}) = d = ab = \phi([\tilde{a}]_{\tilde{G}_0})\,\phi([\tilde{b}]_{\tilde{G}_0}) = \phi([\tilde{a}]_{\tilde{G}_0}[\tilde{b}]_{\tilde{G}_0}) = \phi([\tilde{a}\tilde{b}]_{\tilde{G}_0})$$

$\Rightarrow\ [\tilde{d}]_{\tilde{G}_0} = [\tilde{a}\tilde{b}]_{\tilde{G}_0}$ (for ϕ is an isomorphism $\Rightarrow$ bijective)

$\Rightarrow \exists c \in \tilde{G}_0 : \tilde{a}\tilde{b} = c\tilde{d}$ (for they are the same equivalent class)

↳ depends on a and b

Finally $\underline{U(a)U(b)} = \tilde{U}(\tilde{a})\tilde{U}(\tilde{b}) = \tilde{U}(\tilde{a}\tilde{b}) = \tilde{U}(c\tilde{d}) = \tilde{U}(c)\tilde{U}(\tilde{d}) \cancel{= \sigma(c)U(\tilde{d})}$

$= \sigma(c)\tilde{U}(\tilde{d}) = \sigma(c)U(d) = \underline{\sigma(c)U(ab)}\ \ \exists\, \sigma(c) \in \mathbb{C}^*$ □

Remark: If $\tilde{U}$ is unitary then U is also unitary.

Question: Can we always do this construction? No

but if $\tilde{G}$ exists ($\Leftrightarrow$ any proj. rep. of G is induced by a lin. rep. of $\tilde{G}$)

then we call $\tilde{G}$ the UNIVERSAL COVER of G.

(Important in many places)

Remark: If ~~f~~ G is finite, a universal cover exists.

The universal cover of $SO(3)$ is $SU(2)$.

We are going to constract the universal cover of finite groups.

Observe that if $U(a)U(b) = \omega(a,b)U(ab)$

and $a=e$ or $b=e$, then we know $\omega(a,e) = \omega(e,b) = 1$ ①

Also $U(a)U(b)U(c) = \omega(a,b)U(ab)U(c) = \omega(a,b)\omega(ab,c)U(abc)$

↳ $= U(a)\omega(b,c)U(bc) = \omega(a,bc)\omega(b,c)U(abc)$

$\Rightarrow \omega(a,b)\omega(ab,c) = \omega(a,bc)\omega(b,c)$ (2-cocycle relation) ②

Def. A 2-COCYCLE on G is map $\omega : G \times G \mapsto \mathbb{C}^*$ satisfying ①②.

$\omega \sim \omega'$ (EQUIVALENCE) if $\exists \rho : G \mapsto \mathbb{C}^* : \omega'(a,b) = \frac{\rho(a)\rho(b)}{\rho(ab)}\omega(a,b)$

and we denote by $[\omega]$ the equivalence class containing ω.

We can check $[\omega][\ddot{\omega}] := [\omega\ddot{\omega}]$ defines a PRODUCT ON THE EQUIVALENCE CLASSES of 2-cocycles on G.

Prop. 1) $\{[\omega]\}$ with the above multiplication is a Abelian group, denoted by $M(G)$

called the SCHUR MULTIPLIER or the SECOND COHOMOLOGY GROUP $H^2(G, \mathbb{C}^*)$.

2) If G is finite, then $M(G)$ is also finite, and

$\forall [\omega] \in M(G)\ \exists\, \underline{\omega} \in [\omega], k \in \mathbb{N} : \underline{\omega}(a,b) \in \{e^{i2\pi n/k} \mid n \in \{0, 1, \cdots, k-1\}\}\ \forall a,b \in G$

↳ called the ORDER of $[\omega]$.

Proof as exercise