## II.4 Tensor Product of Representations

Let $\mathcal{H}_1, \mathcal{H}_2$ 2 Hilbert space, let $\varphi_1 \in \mathcal{H}_1$; $\varphi_2 \in \mathcal{H}_2$

Set $\varphi_1 \otimes \varphi_2 : \mathcal{H}_1 \times \mathcal{H}_2 \mapsto \mathbb{C}$, $\varphi_1 \otimes \varphi_2 (\psi_1, \psi_2) := \langle \psi_1, \varphi_1 \rangle_{\mathcal{H}_1} \langle \psi_2, \varphi_2 \rangle_{\mathcal{H}_2}$

Then $\varphi_1 \otimes \varphi_2$ is a bi-antilinear map.

($\Leftrightarrow \varphi_1 \otimes \varphi_2 (\psi_1 + \lambda \psi_1', \psi_2) = \varphi_1 \otimes \varphi_2 (\psi_1, \psi_2) + \bar{\lambda} \varphi_1 \otimes \varphi_2 (\psi_1', \psi_2)$ and same for $\psi_2$)

Let $\varepsilon$ be the set of finite linear combination of $\varphi_1 \otimes \varphi_2$ $(= \{ \sum_{j=1}^{N} \lambda_j \varphi_{1,j} \otimes \varphi_{2,j} \mid N \in \mathbb{N}^*, \lambda_j \in \mathbb{C} \})$

(Observe $\varepsilon$ is a Hilbert space)

and define the scalar product $\langle \varphi_1 \otimes \varphi_2, \varphi_1' \otimes \varphi_2' \rangle_\varepsilon = \langle \varphi_1, \varphi_1' \rangle_{\mathcal{H}_1} \langle \varphi_2, \varphi_2' \rangle_{\mathcal{H}_2}$

Lemma: $\langle \bullet, \bullet \rangle_\varepsilon$ is well-defined and positive definite.

$\hookrightarrow :\Leftrightarrow \langle \varphi_1 \otimes \varphi_2, \varphi_1 \otimes \varphi_2 \rangle_\varepsilon \geq 0$

Def. $\boxed{\mathcal{H}_1 \otimes \mathcal{H}_2}$ is the completion of $\varepsilon$ with respect to the norm associated with $\langle \cdot, \cdot \rangle_\varepsilon$.

$\hookrightarrow$ called the tensor product of $\mathcal{H}_1$ with $\mathcal{H}_2$.

Lemma: If $\{\varphi_{1,j}\}_j$ & $\{\varphi_{2,k}\}_k$ are orthonormal bases of $\mathcal{H}_1$ & $\mathcal{H}_2$, respectively, then $\{\varphi_{1,j} \otimes \varphi_{2,k}\}_{j,k}$ is an orthonormal basis of $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Let $A_j \in \mathcal{B}(\mathcal{H}_j)$, and set $A_1 \otimes A_2 \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ with → Exercise: Proove it's bounded

$(A_1 \otimes A_2)(\varphi_1 \otimes \varphi_2) := A_1 \varphi_1 \otimes A_2 \varphi_2$ and then by linearlity we define $A_1 \otimes A_2$ on any element of $\mathcal{H}_1 \otimes \mathcal{H}_2$.

If $A_1 \otimes A_2$, $B_1 \otimes B_2 \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H}_2)$ Then $(A_1 \otimes A_2)(B_1 \otimes B_2) = A_1 B_1 \otimes A_2 B_2$

and if $\mathcal{H}_1$ & $\mathcal{H}_2$ are finite dimensional then $\mathrm{tr}(A_1 \otimes A_2) = \mathrm{tr}_{\mathcal{H}_1}(A_1)\, \mathrm{tr}_{\mathcal{H}_2}(A_2)$

Let $(G, U, \mathcal{H})$ $(G', U', \mathcal{H}')$ be 2 representations of 2 groups

For $(a, a') \in G \otimes G'$ we set $\mathcal{U}(a, a') = U(a) \otimes U(a') \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H}')$

Then $(G \otimes G, \mathcal{U}, \mathcal{H} \otimes \mathcal{H}')$ is a linear representation

And if $\mathcal{H}, \mathcal{H}'$ are finite dimensional, $\chi_{\mathcal{U}}((a, a')) = \chi_{U}(a)\, \chi_{U}(a')$

Prop. If $(G, U, \mathcal{H})(G', U', \mathcal{H}')$ are irreducible representations of finite groups then

1) $(G \otimes G', \mathcal{U}, \mathcal{H} \otimes \mathcal{H}')$ ~~n~~is irreducible.

2) All irreducible rep. of $G \otimes G$ is of th~~e~~is form.

Consider now the rep of a single group.

Let $G \ni a \overset{\mathcal{U}:}{\mapsto} U(a) \otimes U'(a) \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H}')$

This defines a rep of G, and if $U, U'$ are irreducibe then $(\mathcal{U}, \mathcal{H} \otimes \mathcal{H}')$ might be reducible

If $(\mathcal{U},\mathcal{H})$ $(\mathcal{U}',\mathcal{H}')$ are finite dim, then since $\chi_{\mathcal{U}}(a) = \chi_{u}(a)\ \chi_{u'}(a)$ one gets $\mathcal{U}\otimes\mathcal{U}'$ is equivalent to $\mathcal{U}'\otimes\mathcal{U}$. The decomposition of $\mathcal{U} = \bigoplus_k \nu_k\, \mathcal{U}^k$ can be computed with $\nu_k = \frac{1}{g}\sum_a \overline{\chi_u(a)\,\chi_{u'}(a)}\ \chi^k(a)$

Remark

Consider 2 irreducible representation $(\mathcal{U}^k,\mathcal{H}^k)(\mathcal{U}^j,\mathcal{H}^j)$

Since $\mathcal{H}^j\otimes\mathcal{H}^k = \bigoplus_\ell \nu_\ell \mathcal{H}^\ell$ and $\mathcal{U}^j\otimes\mathcal{U}^k = \bigoplus_\ell \nu_\ell\, \mathcal{U}^\ell$

One can express a suitable basis of $\bigoplus_\ell \nu_\ell \mathcal{H}^\ell$ in terms of the basis $\{e_r^j\otimes e_s^k\}_{r,s}$

The coeffecients relate to the change of basis are called the

CLEBSCH-GORDAN COEFFICIENTS.

Selection Rules

Let $G$ be a group and $(\mathcal{U},\mathcal{H})$ one representation.

Let $\mathcal{U}: G\mapsto \mathrm{Aut}(\mathcal{B}(\mathcal{H}))$ (automorphism 自同構)

with $\mathcal{U}(a)T := \mathcal{U}(a)\,T\,\mathcal{U}(a)^{-1}$ ~~for~~ $\forall\, T\in\mathcal{B}(\mathcal{H})$

one has $\mathcal{U}(ab) = \mathcal{U}(a)\mathcal{U}(b)$, $\mathcal{U}(e)T = \mathcal{U}(e)\,T\,\mathcal{U}(e)^{-1} = T$

We have a linear rep of $G$ on a vector space $\mathcal{B}(\mathcal{H})$

⚠ $\mathcal{B}(\mathcal{H})$ is not a Hilbert space!

If $\mathcal{H}$ is $\underset{\Leftrightarrow \mathcal{H}=\mathbb{C}^n}{\text{finite}}$ then $\underset{=\mathbb{C}^{n\times n}}{\mathcal{B}(\mathcal{H})}$ is of finite dimension $n^2$ and then

$\mathcal{B}(\mathcal{H})$ can be decomposed $= \bigoplus_k \nu_k \mathcal{L}^k$ and $\mathcal{U} = \bigoplus_k \nu_k\, \mathcal{U}^k$

is the decomposition into irreducible rep. with $(\mathcal{U}^k,\mathcal{L}^k)\in \mathfrak{r}^k = [(\mathcal{U}^k,\mathcal{H}^k)]$

It means $\exists\,\phi: \mathcal{H}^k\to\mathcal{L}^k$ bijective : $\phi(\mathcal{U}^k(a)f) = \mathcal{U}^k(a)\,\phi(f) = \mathcal{U}(a)\underbrace{\phi(f)}_{\in\mathcal{L}^k}\mathcal{U}(a)^{-1}$

Thus we have a decomposition of $\mathcal{B}(\mathcal{H})$

which is based on irreducible rep. of $G$.

Thm (Selection Rule)

Let $(G,\mathcal{U},\mathcal{H})$ be a unitary rep and consider $\mathcal{H} = \bigoplus_k \nu_k \mathcal{H}^k$, $\mathcal{U} = \bigoplus_k \nu_k\, \mathcal{U}^k$

Let $(G,\mathcal{U}^j,\mathcal{H}^j)$ be one irred. rep. of $G$ and

Let $\phi: \mathcal{H}^j\mapsto\mathcal{B}(\mathcal{H})$ with $\phi(\mathcal{U}^j(a)f_j) = \mathcal{U}(a)\,\phi(f_j)\,\mathcal{U}(a)^{-1}$ $\forall f_j\in\mathcal{H}^j$

Then $\forall f_k\in\mathcal{H}^k\subset\mathcal{H}\ \forall f_i\in\mathcal{H}^i\subset\mathcal{H}$: $\langle f_i, \phi(f_j) f_k\rangle = 0$

EXCEPT if $(\mathcal{H}^i,\mathcal{U}^i)$ appears in the decomposition of $(\mathcal{H}^j\otimes\mathcal{H}^k)$ & $(\mathcal{U}^j\otimes\mathcal{U}^k)$

in a sum of irred. rep.

Remark: This result is also related to the Clebsch-Gordan coef. (Wigner-Eckart Thm.)

## II.5 Symmetries and Projective Representation

Consider $\mathcal{H}$ a Hilbert space, and $\hat{\mathcal{H}} := \mathcal{H}/\mathbb{C}$

which means $\hat{\mathcal{H}} \ni \hat{\varphi} = \{ \psi \in \mathcal{H} \mid \varphi = \alpha \psi \; \exists \alpha \in \mathbb{C} \}$

This interest of $\hat{\mathcal{H}}$ is that its elements are in bijection with all pure states, which means with all 1D prejection of the form

$$P_{\hat{\varphi}} = |\varphi\rangle\langle\varphi| \text{ with } \varphi \in \hat{\varphi} \text{ and } \|\varphi\| = 1.$$

For 2 such pure states $P_{\hat{\psi}}$ and $P_{\hat{\varphi}}$

the transition probabily is defined by

$$\mathrm{Tr}(P_{\hat{\psi}} P_{\hat{\varphi}}) = |\langle \varphi, \psi \rangle|^2 \text{ with } \varphi \in \hat{\varphi}, \; \psi \in \hat{\psi}, \; \|\varphi\| = 1 = \|\psi\|$$

Def. A SYMMETRY is a map

$$S : \hat{\mathcal{H}} \mapsto \hat{\mathcal{H}} \text{ satisfying}$$

$$\mathrm{Tr}(P_{S\hat{\psi}} P_{S\hat{\varphi}}) = \mathrm{Tr}(P_{\hat{\psi}} P_{\hat{\varphi}})$$