Let $(U^k, \mathcal{H}^k)$ be a unitary and irreducible representation of G,

and $\{\gamma^k\}_k$: with $\gamma^k := \underbrace{[(U^k, \mathcal{H}^k)]}_{\text{equivalence class}}$ be an enumeration of such rep.

Let $\{e_j^k\}_{j=1}^{n_k}$ be $\underbrace{\text{a}}_{\text{orthonormal}}$ basis of $\mathcal{H}^k$. with $n_k = \dim \mathcal{H}^k$

Let us set

$U_j^k(a) = \langle e_i^k, U^k(a) e_j^k \rangle$

and $(U_{ij}(a))_{a\in G} \in \ell^2(G) \longrightarrow := \{\overbrace{(a_1, \cdots, a_{|G|})}^{a}\,|\, a_j \in \mathbb{C}, \sum_{j=1}^{|G|} |a_j|^2 < \infty\}$

$\ell^2(G)$ is a $\mathcal{H}$ with $\langle a, b\rangle = \sum_{j=1}^{|G|} \overline{a_j} b_j$. $\dim \ell^2(G) = |G| =: g$

Consider $T := |e_s^l\rangle\langle e_j^k| \in \mathcal{B}(\mathcal{H}^k, \mathcal{H}^l)$

If $l \neq k$ then $Z_T = 0$

$\Rightarrow 0 = \langle e_r^l, Z_T e_i^k\rangle = \frac{1}{g}\sum_a \langle e_r^l, U^l(a) e_s^l\rangle \langle e_j^k, U^k(a)^{-1} e_i^k\rangle$

$= \frac{1}{g}\sum_a U_{rs}^l(a)\, \overline{U_{ij}^k(a)}$

$\Rightarrow (U_{ij}^k(a))_{a\in G} \perp (U_{rs}^l(a))_{a\in G}$

If $l = k$ then $Z_T = \frac{1}{n_k}\operatorname{tr}(|e_s^k\rangle\langle e_j^k|)\mathbb{1} = \frac{1}{n_k}\delta_{sj}\mathbb{1}$

$\Rightarrow \frac{1}{g}\sum_a U_{rs}^l(a)\overline{U_{ij}^k(a)} = \langle e_r^k, Z_T e_i^k\rangle = \frac{1}{n_k}\delta_{sj}\delta_{ri}$

In summary, $\frac{1}{g}\sum_a U_{rs}^l(a)\, U_{ij}^k(a) = \frac{1}{n_k}\delta_{kl}\delta_{sj}\delta_{ri}$ (*)

Corollary: $\exists N < \infty$ inequivalent unitary irreducible representations of G

with $\sum_{k=1}^{N} r_k^2 = g$

Proof of $\leq$

For each rep. $(U^k, \mathcal{H}^k)$ one has n_k^2 elements of $\ell^2(G)$ which are orthogonal $\rightarrow$ of dimension g

$\Rightarrow \sum_{k=1}^{N} r_k^2 \leq \dim \ell^2(G) = g$ □

For any finite dimensional representation of G we set $\chi(a) := \operatorname{tr}(U(a))$

$\{\chi(a)\}_{a\in G}$ is called the set of the characters of G. in $\mathcal{H}$.

Again $\chi(\cdot) \in \ell^2(G)$

Corollary: Let $(U^k, \mathcal{H}^k)$ $(U^l, \mathcal{H}^l)$ be 2 unitary irreducible rep of G then

$$\frac{1}{g}\sum_a \chi^l(a)\,\overline{\chi_k(a)} = \begin{cases} 1 & \text{if } (U^k, \mathcal{H}^k) \sim (U^l, \mathcal{H}^l) \\ 0 & \text{otherwise} \end{cases}$$

Proof

Observe that the character depends only on the equivalent class of rep.

and not on the representative.

Since $\chi^k(a) = \sum_{j=1}^{n_k} U^k_{jj}(a)$ one has

$$\frac{1}{g}\sum_{a\in G}\chi^\ell(a)\overline{\chi^k(a)} = \underbrace{\frac{1}{g}\sum_{a\in G}\sum_{j=1}^{n_k}\sum_{r=1}^{n_\ell} U^\ell_{rr}(a)\overline{U^k_{jj}(a)}}_{\frac{1}{n_k}\delta_{k\ell}\delta_{rj}\delta_{rj}} = \sum_{j=1}^{n_k}\frac{1}{n_k}\delta_{k\ell} = \delta_{k\ell} \qquad \square$$

If $(U,\mathcal{H})$ is a completely reducible and finite dim rep of G, then

$$\mathcal{H} = \bigoplus_{k=1}^{N} \nu_k \mathcal{H}^k \quad , U = \bigoplus_{k=1}^{N}\nu_k U^k \qquad U = \begin{pmatrix} U_1 & & & \\ & U_1 & & \\ & & U_2 & \\ & & & U_5 \end{pmatrix} \quad \begin{array}{l} \nu_1 = 2 \\ \nu_2 = \nu_5 = 1 \\ \nu_3 = \nu_4 = 0 \end{array}$$

$\uparrow$ modulo rearrangement $\quad \hookrightarrow \in \mathbb{N}$ $\quad \downarrow$ multiplicity of the representation of type η^k

Thm. Let $(U,\mathcal{H})$ be a finite dim rep of G, then

1) $\nu_k := \frac{1}{g}\sum_{a\in G}\overline{\chi(a)}\,\chi^k(a)$ ← character in rep $\mathcal{H}^k$

2) This rep is irreducible iff $\frac{1}{g}\sum|\chi(a)|^2 = 1$

3) If $(U',\mathcal{H}')$ is another finite dim rep of G, then $(U,\mathcal{H})\sim(U,\mathcal{H})$ iff their χ are equal.

We introduce the regular representation of G

Def. Let G be finite group and set $\mathcal{H}^{reg} := \ell^2(G)$

and $[U^{reg}(a)f](b) = f(a^{-1}b)$ for $f\in\mathcal{H}^{reg}$

Exercise: Check if it is a representation.

$(U^{reg},\mathcal{H}^{reg})$ is called the regular representation.

This representation is completely reducible since G is finite.

$\Rightarrow \mathcal{H}^{reg} = \bigoplus_{k=1}^{N}\nu_k\mathcal{H}^k$, $U^{reg} = \bigoplus_{k=1}^{N}\nu^k U^k$ and $\sum_{k=1}^{N}\nu_k n_k = g = \dim\mathcal{H}^{reg}$

Let us define $\delta_a \in \ell^2(G)$ by $\delta_a(b) = 1$ if $a=b$ and $\delta_a(b)=0$ otherwise

and $\{\delta_a\}_{a\in G}$ is an orthonormal basis of $\mathcal{H}^{reg}$

One has $U^{reg}_{bc}(a) = \langle\delta_b, U^{reg}(a)\delta_c\rangle = \langle\delta_b, \delta_c(a^{-1}\bullet)\rangle = \sum_{d\in G}\delta_b(d)\delta_c(a^{-1}d) = \delta_c(a^{-1}b) = \begin{cases}1 & \text{if } c=a^{-1}b\\ 0 & \text{otherwise}\end{cases}$

In particular if $b=c$, $U^{reg}_{bb}(a) = \begin{cases}1 & \text{if } b=a^{-1}b\\ 0 & \text{otherwise}\end{cases} = \begin{cases}1 & \text{if } a=e\\ 0 & \text{otherwise}\end{cases} \Rightarrow \chi^{reg}(a) = \begin{cases} g & \text{if } a\in e\\ 0 & \text{otherwise}\end{cases}$

Thm. 1) $\sum_{k=1}^{N} n_k^2 = g$

2) $U^{reg} = \bigoplus_{k=1}^{N} n_k U^k$

$\hookrightarrow$ multiplicity $= \dim\mathcal{H}^k$ (only true for regular rep)

Proof: $\nu_k = \frac{1}{g}\sum_{a\in G}\overline{\chi^{reg}(a)}\,\chi^k(a) = \frac{1}{g}\,g\,\chi^k(e) = n_k \qquad \square$

$\Rightarrow$ The regular representation contains all irreducible representations.

Lemma

Let $C^1, \cdots, C^M$ be the list of conjugacy classes of finite group G with $d_\ell := |C^\ell|$. Then

$$\frac{1}{g} d_\ell \sum_{k=1}^{N} \overline{\chi^k(C^\ell)} \chi^k(C^i) = \delta_{i\ell}$$

$\rightarrow$ character on any element of C^ℓ (same)

Thm. For any finite group $N = M$

Proof:

$$\sum_{\ell=1}^{M} \frac{1}{g} d_\ell \sum_{k=1}^{N} |\chi_k(C^\ell)|^2 = M$$

Also

$$\sum_{a \in G} = \sum_{\ell=1}^{M} \sum_{a \in C^\ell}, \text{ then}$$

$$N = \sum_{k=1}^{N} \frac{1}{g} \sum_{a \in G} \overline{\chi^k(a)} \chi^k(a) = \sum_{k=1}^{N} \frac{1}{g} \sum_{\ell=1}^{M} \sum_{a \in C^\ell} |\chi^k(a)|^2 = \sum_{k=1}^{N} \frac{1}{g} \sum_{\ell=1}^{M} d_\ell |\chi^k(C^\ell)|^2$$

$$= \sum_{\ell=1}^{M} \frac{1}{g} d_\ell \sum_{k=1}^{N} |\chi^k(C^\ell)|^2 = M \qquad \square$$