II. Linear representation

Def. A bounded linear operator is $\overbrace{\text{a map}}^{\text{linear}}$ $T: \mathcal{H} \to \mathcal{H}$ s.t.

$\exists c < \infty\ \forall f \in \mathcal{H} : \|Tf\| \leq c\|f\|$

The infimum over c is called the NORM of $T =: \|T\|$

$B(\mathcal{H})$ ($\equiv M_n(\mathbb{C})$) := the set of all b. l. op. on $\mathcal{H}$. (It is a group).

Def. Linear map T^* satisfying $\langle T^*f, g\rangle = \langle f, Tg\rangle\ \forall f, g \in \mathcal{H}$

is called the ADJOINT of T. (always exist)

*In finite dimension, $[T^*] = \overline{[T^t]}$ → complex conjugate → transposition

Def. If $\langle f, Tf\rangle \geq 0\ \forall f \in \mathcal{H}$, T is POSITIVE. (then $T^* = T$)

Def. Let $T \in B(\mathcal{H})$

1) T is UNITARY if $T^*T = TT^* = \mathbb{1}$.

2) T is an ORTHOGONAL PROJECTION if $T^2 = T = T^*$ (T^2 → $T \cdot T$)

3) T is INVERTIBLE (in $B(\mathcal{H})$) if

$T: \mathcal{H} \to \mathcal{H}$ is bijective $\Leftrightarrow \exists T^{-1} \in B(\mathcal{H}) : TT^{-1} = T^{-1}T = \mathbb{1}$

For linear representation

Def. Let G be a group and $\mathcal{H}$ a hilbert space.

A [LINEAR] REPRESENTATION of G in $\mathcal{H}$ is a hormomorphism

$U: G \mapsto B(\mathcal{H})$

it means $U(ab) = U(a)U(b)$ and $U(e) = \mathbb{1} \Rightarrow U(a^{-1}) = U(a)^{-1}$

Remark: This definition can be generized to non-linear or to projective, …, representati

If $U(a)$ is unitary, $\forall a \in G$, we speak about a unitary representation.

$U: G \to \mathcal{U}(\mathcal{H}) \subset B(\mathcal{H})$

Def. · $U: G \mapsto B(\mathcal{H})$ is TRIVIAL if $U(a) = \mathbb{1}\ \forall a \in G$

· $U: G \mapsto B(\mathcal{H})$ is FAITHFUL if $U(a) \neq \mathbb{1}\ \forall a \in G \backslash \{e\} \Rightarrow$ injective

· The DIMENSION of the representation is the dimension of $\mathcal{H}$.

Lemma $U: G \mapsto B(\mathcal{H})$.

1) If $G_0 \triangleleft G$, and if $V_0: G/G_0 \mapsto B(\mathcal{H})$ is a representation

Then $V(a) := ([a]_{G_0})$ defines a representation of G in $\mathcal{H}$ $(a \in G)$

2) The set $G_0 := \{a \in G \mid U(a) = \mathbb{1}\} \triangleleft G$.

3) In particular if G is simple, then all non-trivial representations are faithful.

see lecture 1

Def. Let G be a group, $U: G \mapsto \mathcal{B}(\mathcal{H})$ and $U': G \mapsto \mathcal{B}(\mathcal{H}')$ 2 representations of G.
They are SIMILAR or EQUIVALENT if
$\exists S: \mathcal{H} \mapsto \mathcal{H}'$ a linear operator which is bijective and s.t.
$U'(a) = S \cdot U(a) \cdot S^{-1} \ \forall a \in G$.
They are UNITARILY EQUIVALENT if in addition
$S^* = S^{-1}$

Thm. Let G be finite and $U: G \mapsto \mathcal{B}(\mathcal{H})$ be a linear representation
Then U is similar to a unitary representation $U': G \mapsto \underbrace{\mathcal{U}(\mathcal{H})}_{\text{set of unitary elements in } \mathcal{H}}$

II.2 Reducible or Irreducible Representation

Recall if $\mathcal{H}_0$ is a closed subspace of $\mathcal{H}$, there exists $\mathcal{H}_1$ also closed subspace of $\mathcal{H}$ such that $\mathcal{H}_0 \underbrace{\oplus}_{\text{orthogonal sum}} \mathcal{H}_1 = \mathcal{H}$

If $T \in \mathcal{B}(\mathcal{H})$ then T can be written as

$$\begin{pmatrix} T_{00} & T_{01} \\ T_{10} & T_{11} \end{pmatrix}$$

(always true for matrices, but be careful when ~~infeinitely~~ infinitely dimensional)

Def. Let $(G, U, \mathcal{H})$ be a group and a linear representation.
- A closed subspace $\mathcal{H}_0 \subset \mathcal{H}$ is INVARIANT under the representation if
$U(a)\mathcal{H}_0 \subset \mathcal{H}_0$ ($\Leftrightarrow \forall f \in \mathcal{H}_0 : U(a) f \in \mathcal{H}_0$) $\exists a \in G$
- $\mathcal{H}_0$ is PROPER if $\mathcal{H}_0 \neq \mathcal{H}$ and NON-TRIVAL if $\mathcal{H}_0 \neq \{0\}$
- $\mathcal{H}_0$ is MINIMAL if $\nexists \mathcal{H}_1 : \{0\} \neq \mathcal{H}_1 \subsetneq \mathcal{H}_0$ with $\mathcal{H}_1$ invariant
- $(U, \mathcal{H})$ ($\equiv$ the linear representation) is IRREDUCIBLE if
$\{0\}$ and $\mathcal{H}$ are the only invariant closed subspaces.
($\equiv \mathcal{H}$ is minimal) Otherwise it is REDUCIBLE.

Lemma: If G is finite, and $(U, \mathcal{H})$ is an irreducible representation of G, then $\dim \mathcal{H} \leq |G|$. Proof as exercise

Observasion: Consider $(G, U, \mathcal{H})$ and $\mathcal{H}_0 \subset \mathcal{H}$ invariant, then

$$\forall a \in G : U(a) = \begin{pmatrix} U(a)_{00} & U(a)_{01} \\ 0 & U(a)_{11} \end{pmatrix} \text{ in } \mathcal{H}_0 \oplus \mathcal{H}_0^\perp$$

↱ not implying the existance of $\mathcal{H}_0$

Def. $(G, U, \mathcal{H})$ is COMPLETELY REDUCIBIE if $\forall \mathcal{H}_0 \subset \mathcal{H}$ invariant one has

$$U(a) = \begin{pmatrix} U(a)_{00} & 0 \\ 0 & U(a)_{11} \end{pmatrix} \forall a \in G \quad (\Leftrightarrow \mathcal{H}_0^{\perp} \text{ is invariant too})$$

Thm: 1) If $(G, U, \mathcal{H})$ is a unitary representation then it's completely reducible.

2) If G is finite then $(G, U, \mathcal{H})$ is completely reducible.

Shrur Lemma

Let $(G, U, \mathcal{H})$ be a finite dimensional irreducible rep of G,

$(U', \mathcal{H}')$ be another (maybe $\dim = \infty$) irreducible rep of G.

Let $Z: \mathcal{H} \to \mathcal{H}'$ be linear satisfying $ZU(a) = U'(a)Z \ \forall a \in G$

Then either $Z = 0$ or Z is a similarity transformation. ($\Rightarrow U$ and U' are similar)

Corollory

1) Let $(G, U, \mathcal{H})$ be a finite dimensional irreducible rep of G

Let $T \in \mathcal{B}(\mathcal{H})$ s.t. $U(a)T = TU(a) \ \forall a \in G$

Then $T = \lambda \mathbb{1} \ \exists \lambda \in \mathbb{C}$. (and λ is the eigenvalue of T)

Proof

Since $\dim \mathcal{H} < \infty$, T is a matrix and has at least one eigenvalue λ.

Then $(T - \lambda \mathbb{1}) U(a) = U(a)(T - \lambda \mathbb{1}) \ \forall a \in G$ ↳ $\Rightarrow (T - \lambda \mathbb{1})v = 0$ $\Rightarrow$ not inversible

By Shrur Lemma, since $T - \lambda \mathbb{1}$ cannot be a similarity transformation, (since $T - \lambda \mathbb{1}$ is not inversible), one has

$T - \lambda \mathbb{1} = 0$ □

2) If G is abelian, any finite dimensional irreducible rep of G is of dimension 1. (Proof as exercise)

Prop: Let G be a finite group and G_0 an abelian subgroup.

Then any finite dim. irreducible rep. of G is of dimension $\leq \frac{|G|}{|G_0|}$

II.3 Representations of Finite Group

Prop. $(U,\mathcal{H})$ $(U',\mathcal{H})$ be 2 lin. rep. of a finite group G, and let

$T:\mathcal{H} \mapsto \mathcal{H}'$ bounded

$$Z_T := \frac{1}{|G|}\sum_{a\in G} U'(a)\,T\,U(a)^{-1}$$

Then

1) If $(U,\mathcal{H}) \overset{\text{not similar}}{\not\sim} (U',\mathcal{H}')$ then $Z_T = 0$

2) If $(U,\mathcal{H}) = (U',\mathcal{H}')$, then $Z_T = \frac{1}{n}\overset{\text{trace}}{\mathrm{tr}}(T)\,\mathbb{1}$ with $n = \dim\mathcal{H}$.

Proof: 1) One checks $U'(b)Z_T = Z_T U(b)$ $\forall b\in G$

By Schur Lemma $\Rightarrow Z_T = 0$.

2) By the previous corollary, $Z_T = \lambda\mathbb{1}$. Then

$$\mathrm{tr}(Z_T) = \lambda\,\mathrm{tr}(\mathbb{1}) = \lambda n$$

$$\|$$

$$\mathrm{tr}\Big(\frac{1}{|G|}\sum_{a\in G} U(a)\,T\,U(a)^{-1}\Big) = \frac{1}{|G|}\sum_{a\in G}\mathrm{tr}(T\,\cancel{U(a)^{-1}}\,\cancel{U(a)}) = \frac{1}{|G|}\sum_{a\in G}\mathrm{tr}(T) = \mathrm{tr}(T)$$

$$\Rightarrow \lambda = \frac{\mathrm{tr}(T)}{n} \Rightarrow Z_T = \frac{1}{n}\mathrm{tr}(T)\,\mathbb{1}$$ $\square$