Aim: find all finite subgroups of $O(3)$

$\forall n \in O_j \subset X$, the number of $R(\neq \mathbb{1})$ which leaves n invariant is $g_j - 1$

Let $p_j := |O_j|$. Then $\sum_{j=1}^{r} p_j (g_j - 1) = 2(g-1)$ (☆)

↳ # of elements of S^2 invariant by ≥ 1 element of G, multiplicity counted

$\Leftrightarrow \sum_{j=1}^{r} g - p_j = 2(g-1)$

$\Leftrightarrow rg - \sum_{j=1}^{r} p_j = 2(g-1)$

$\Leftrightarrow 2 - \frac{2}{g} = r - \sum_{j=1}^{r} \frac{1}{g_j}$ # (*)

Since $g_j \geq 2$, $\sum_{j=1}^{r} \frac{1}{g_j} \leq \frac{r}{2} \Leftrightarrow -\sum_{j=1}^{r} \frac{1}{g_j} \geq -\frac{r}{2}$

$\Rightarrow 2 - \frac{2}{g} = r - \sum_{j=1}^{r} \frac{1}{g_j} \geq r - \frac{r}{2} = \frac{r}{2}$

$\Rightarrow r < 4$. At most X is decomposed to 3 orbits.

Also: $r > 1$. Since otherwise $2 - \frac{2}{g} = 1 - \frac{1}{g_1} \Leftrightarrow \frac{1}{g_1} = \frac{2}{g} - 1 \leq 0$

$\Rightarrow r \in \{2, 3\}$

If $r = 2$: $2 - \frac{2}{g} = 2 - \frac{1}{g_1} - \frac{1}{g_2} \Leftrightarrow \frac{2}{g} = \frac{1}{g_1} + \frac{1}{g_2}$ (*)

We have $g_j \leq g \Rightarrow g_1 = g_2 = g$

$\Rightarrow$ Each element of X is invariant under all elements of G

$\Rightarrow G \simeq C_g$ = Cyclic group: rotations by $\frac{2\pi}{g}k$ for $k \in \{0, \cdots, g-1\}$

If $r = 3$: (*) $\Leftrightarrow 2 - \frac{2}{g} = 3 - \frac{1}{g_1} - \frac{1}{g_2} - \frac{1}{g_3} \Leftrightarrow \frac{1}{g_1} + \frac{1}{g_2} + \frac{1}{g_3} = 1 + \frac{2}{g}$

Wlog (Without loss of generality) $g_1 \leq g_2 \leq g_3 \Rightarrow \frac{1}{g_1} \geq \frac{1}{g_2} \geq \frac{1}{g_3}$

If $g_1 > 2 (\Leftrightarrow g_1 \geq 3) \Rightarrow \frac{1}{g_1} + \frac{1}{g_2} + \frac{1}{g_3} \leq \frac{3}{g_1} \leq 1 \Rightarrow g < 0$ contradiction

$\Rightarrow g_1 = 2$. (*) $\Leftrightarrow \frac{1}{g_2} + \frac{1}{g_3} = \frac{1}{2} + \frac{2}{g}$

If $g_2 \geq 4 \Rightarrow \frac{1}{g_2} + \frac{1}{g_3} \leq \frac{2}{g_2} \leq \frac{1}{2} \Rightarrow g < 0$ contradiction again

$\Rightarrow g_2 \in \{2, 3\}$.

Similarly, if $g_2 = 3$, then $g_3 \leq 5$

$\Rightarrow$ All possibilities of (g_1, g_2, g_3):

$(2, 2, l)$ for $l \in \mathbb{N}_+ \setminus \{1\}$; $\Rightarrow g = 2l$ group D_l (dehedral)

$(2, 3, 3)$ $\Rightarrow g = 12$ group T (tetrahedral)

$(2, 3, 4)$ $\Rightarrow g = 24$ group O (octahedral)

$(2, 3, 5)$ $\Rightarrow g = 60$ group I (icosahedral)

Lemma: Let G be a cpg and consider $\{R \equiv (n, \frac{2\pi}{l}k) \mid k \in \{0, 1, \cdots, l-1\}\} \subset G$ be subgroup.
Then $l \in \{1, 2, 3, 4, 6\}$

Proof: Compute trace(R) in two different bases.
According to prop of trace, they are equal.

$$\text{In 1 basis } R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\delta & -\sin\delta \\ 0 & \sin\delta & \cos\delta \end{pmatrix} \Rightarrow tr(R) = 1 + 2\cos\delta$$

Recall $\mathcal{L} = \{c_1 b_1 + c_2 b_2 + c_3 b_3 \mid c_j \in \mathbb{Z}\}$ and $b_1 \sim b_3$ 3 lin. indep. elements of $\mathbb{R}^3$

Then $Rb_j = \sum_{k=1}^{3} c_{jk} b_k$ with $c_{jk} \in \mathbb{Z}$

$\Rightarrow tr(R) = c_{11} + c_{22} + c_{33} \in \mathbb{Z}$

$\Rightarrow 1 + 2\cos\delta \in \mathbb{Z} \Rightarrow 2\cos\delta \in \mathbb{Z} \Rightarrow \delta \in \{\frac{2\pi}{l}k \mid k = \{0, \cdots, l-1\}, l \in \{1, 2, 3, 4, 6\}\}$ □

Remark: Same result for matrix $-\mathbb{1}R$ for $R \in SO(3)$

In summary: The cpg of 1st type are

$C_1, C_2, C_3, C_4, C_6, D_2, D_3, D_4, D_6, T, O$

C_1 ↓ trivial group (I has a rotation with $\ell = 5$)

Let us now study the finite subgroups of $O(3)$.

Recall $O(3) = SO(3) \otimes \{\mathbb{1}, -\mathbb{1}\} = SO(3) \sqcup (-\mathbb{1})\, SO(3)$, $(-\mathbb{1}) =: \Pi$

det	± 1	$+1$	-1
	$O(3)$	$SO(3)$	$O(3) \setminus SO(3)$
	$\cap$	$\cap$	$\cap$
	G	G_+	G_-

2 cases: Either $G = G_+ \sqcup G_-$ contains Π or does not contain Π

1) If $\Pi \in G$ then $\Pi \in G_-$ and $G_- = \Pi G_+$ by using $\det(AB) = \det(A)\det(B)$
$\Rightarrow$ 11 new cpg made of $G_+ \sqcup \Pi G_+$ with G_+ in the list of 1st type.
They are called cpg of 2nd type with inversion.

2) If $\Pi \notin G$

Let us set $\phi: G \to SO(3)$, $\phi(R) = \begin{cases} R & \text{if } R \in G_+ \\ \Pi R & \text{if } R \in G_- \end{cases}$ and set $\mathbb{G} := \phi(G)$.

Observe $G_+ \cap \Pi G_- = \emptyset$. (empty set)

(Indeed if $G_+ \ni a = \Pi b$, $b \in G_-$ then $\Pi = ab^{-1} \in G$)

and also ϕ is a homomorphism and an isomorphism between $G \leftrightarrow \mathbb{G}$

Also $|G_+| = |G_-|$ (exercise)

$\Rightarrow \mathbb{G}$ is a finite subgroup of $SO(3)$ containing a subgroup G_+ and a subset $\phi(G_-)$

$\mathbb{G} = G_+ \sqcup \phi(G_-)$, $|\mathbb{G}| = 2g$

By inspection, the possible pairs (G_+, G) are

$$\left.\begin{array}{ll} (C_n, C_{2n}) & \text{for } n\in\{1,2,3\} \\ (C_n, D_{2n}) & \quad '' \\ (D_n, D_{2n}) & \quad '' \\ (T, O) & \end{array}\right\} \text{10 solutions}$$

Implicit: Column 1 are subgroups of column 2

Now we have all 32 finite subgroups of $O(3)$ which leave a lattice invariant

Next step: for a given subgroup, find the corresponding invariant lattice

$\rightarrow$ 7 lattice systems

14 Bravais lattices

Exercise: Do the same thing with $E(3)$ (containing translation) instead of $O(3)$

$\rightarrow$ 230 finite subgroups leaving a lattice invariant.

Or for $O(2)$

Chapter II: Linear representation

II.1 Generalities

Def. A Hilbert space is a complex vector space together with an inner product $\langle\cdot,\cdot\rangle$ (linear in the second argument) and complete for the norm $\|f\| = \sqrt{\langle f, f\rangle}$

Example: ($\mathbb{R}^n$ "real Hilbert space")

- $\mathbb{C}^n$ with $a=(a_1,\cdots,a_n), b=(b_1,\cdots,b_n)\in\mathbb{C}^n$ $\langle a,b\rangle = \sum_{j=1}^{n} \overline{a_j}\, b_j$
- $L^2(\mathbb{R}^n)$ with $\langle f,g\rangle = \int_{\mathbb{R}^n} \overline{f(x)}\, g(x)\, dx$
- $\ell^2(\mathbb{Z}^n)$ with $\langle a,b\rangle = \sum_{j\in\mathbb{Z}^n} \overline{a_j}\, b_j$

Remark: $\langle a,b\rangle = \overline{\langle b,a\rangle}$ and $\langle a,a\rangle \geq 0$ with equality iff $a=\mathbf{0}$