Note: For $m \times m$ matrices A, B,

$\det(AB) = \det(A)\det(B) \Rightarrow$ det is a homomorphism from $GL(n, \mathbb{C})$ to $\mathbb{C}^*$

⚠ $\det(A+B) \neq \det(A) + \det(B)$ generally

Question: How to construct 1 group from 2 groups?

Def. Let G, G' be 2 groups, and set

$G \otimes G' := \{a \otimes a' \mid a \in G, a' \in G'\}$

with $(a \otimes a')(b \otimes b') := ab \otimes a'b'$, $e = e \otimes e'$, $(a \otimes a')^{-1} = a^{-1} \otimes (a')^{-1}$

Then $G \otimes G'$ can be proved to be a group, called the DIRECT PRODUCT of G & G'

Conversely, if G is group and G_1, G_2 are subgroups of G with

1) $G_1 \cap G_2 = \{e\}$

2) $\forall a_j \in G_j : a_1 a_2 = a_2 a_1$

3) $\forall a \in G \; \exists a_1 \in G_1, a_2 \in G_2 : a = a_1 a_2$

Then $G \simeq G_1 \otimes G_2$

Observe this G_1, G_2 is unique (hint: from $G_1 \cap G_2 = \{e\}$)

and G_1, G_2 are normal.

Def. (INNER SEMI-DIRECT PRODUCT)

G is a semi-direct product if $\exists G_1, G_2$ subgroups with

1) G_1 normal 2) $G_1 \cap G_2 = \{e\}$

3) $\forall a \in G$, $a = a_1 a_2$ with $a_j \in G_j$ $\Big\}$ We write $G = G_1 \odot G_2 = G_1 \rtimes G_2$

Abstract (OUTER SEMI-DIRECT PRODUCT)

Let H, N be 2 groups and let

$\phi : H \to \mathrm{Aut}(N)$ be a homomorphism

↳ $= \{$automorphism on $N\}$ Check that $\mathrm{Aut}(N)$ is a group.

Then we set $N \rtimes_\phi H = \{(n, h) \in N \times H\}$ with the product

$(n_1, h_1)(n_2, h_2) = (n_1 \phi(h_1) n_2, h_1 h_2)$ and note that

$e = (e_N, e_H)$, $(n, h)^{-1} = (\phi(h^{-1})(n^{-1}), h^{-1})$

↳ For $h \in H$, $\phi(h) \in \mathrm{Aut}(N) \therefore \phi(h)$ is a map $N \to N$

Also, if we set $G_1 = \{(n, e_H) \mid n \in N\}$ and $G_2 = \{(e_N, h) \mid h \in H\}$

then G_1 is normal in $N \rtimes_\phi H$ and

$G_1 \odot G_2 = N \rtimes_\phi H$.

Prop.

The map $R : SU(2) \to SO(3)$ defined by

$R(U)_{jk} = \frac{1}{2} tr(\sigma_j U \sigma_k U^{-1}) \to$ trace

with $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

is a surjective map with $\ker(R) = \{\mathbb{1}, -\mathbb{1}\}$

(homomorphism)

(typical in quantum mechanics)

Lemma: $O(3) \simeq SO(3) \otimes J$ with $J = \{1, -1\}$

$SL(n, \mathbb{C}) \triangleleft GL(n, \mathbb{C})$

$\exists a$

($\because \det(cac^{-1}) = \det(a)$ $\therefore$ normal)

Back to examples

5) Euclidean group $E(n) = \{(A, b) \mid A \in O(n), b \in \mathbb{R}^n\}$

with $(A,b)(A',b') = (AA', b + Ab')$,

$e = (\mathbb{1}, \mathbf{0})$ and $(A, b)^{-1} = (A^{-1}, -A^{-1}b)$

Observe $E(n) = (\mathbb{1}, \mathbb{R}^n) \rtimes (O(n), \mathbf{0})$ (inner semi-direct product)

Exercise: represent it as an outer semi-direct product (find ϕ)

6) If $g = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \in M_4(\mathbb{R})$, the Lorentz group L is given by

$\equiv M_{4\times4}(\mathbb{R}) \equiv \mathbb{R}^{4\times4}$

$L = \{\Lambda \in M_4(\mathbb{R}) \mid \Lambda^t g \Lambda = g\}$

7) Poincaré group $P = \{(\Lambda, b) \mid \Lambda \in L, b \in \mathbb{R}\}$

with $(\Lambda b)(\Lambda' b') = (\Lambda\Lambda', b + \Lambda b')$

$\Rightarrow P = (\mathbb{1}, \mathbb{R}^4) \rtimes (L, \mathbf{0})$

Def. A group of transformation of a set X is
a set X, a group G, and a map $\circ: G\times X\to X$ s.t.

$$a\circ\underbrace{(b\circ x)}_{\in X} = \underbrace{(ab)}_{\in G}\circ x,\ \text{and}\ e\circ x = x$$

Also called: X is a G-set.

Def. $\forall x\in X$ The set $O(x)\equiv O_x := \{a\circ x \mid a\in G\}\subset X$ called the ORBIT of x.
$S(x)\equiv G_x := \{a\in G \mid a\circ x = x\}\subset G$, called the STABILIZER of x.

Lemma:

1) The set of orbits defines a partition of X.
2) $S(x)$ is a subgroup of G
3) If $x'\in O(x)$ then $S(x)\simeq S(x')$

Lemma:

Let G be a finite group of transformation of X, then
$\forall x\in X: |S(x)|\cdot|O(x)| = |G|$

Remark:

The Euclidean group is the group of transformation of $\mathbb{R}^n$ leaving $\|x-y\|$ invariant
The Poincaré group is " " of $\mathbb{R}^4$
leaving $x^\nu y_\nu := x_1y_1 - x_2y_2 - x_3y_3 - x_4y_4$ invariant.

Remark: $O(3) = SO(3) \sqcup \left(\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right) SO(3)$ ($\sqcup$: disjoint union) $A\sqcup B$ means $A\cup B$ with $A\cap B=\emptyset$

$\forall R\in SO(3)\ \exists n\in\mathbb{R}^3$ with $\|n\|=1$: s.t. $Rn = n$

Then in a basis (n, e_2, e_3) R takes the form

$$\begin{pmatrix} 1 & 0 & 0\\ 0 & \cos(\delta) & -\sin(\delta)\\ 0 & \sin(\delta) & \cos(\delta)\end{pmatrix} \exists\delta\in[0,2\pi)$$

We can parameterize R with n and δ.

I.2 Crystallographic Groups

Def. Let b_1, b_2, b_3 be 3 lin. indep. (linearly independent) vectors in $\mathbb{R}^3$, and set

$$\mathcal{L} = \{m_1 b_1 + m_2 b_2 + m_3 b_3 \mid m_j \in \mathbb{Z}\} \Rightarrow (0,0,0) \in \mathcal{L}$$

$\mathcal{L}$ is called a LATTICE in $\mathbb{R}^3$. (used for discribing a crystal)

Def. A CRYSTALLOGRAPHIC POINT GROUP is a subgroup of $O(3)$ which leaves a lattice $\mathcal{L}$ invariant.

$\forall A \in CPG : A\mathcal{L} = \mathcal{L}$

(We'll find 32 such groups)

Remark: Given a c.p.g. the lattice $\mathcal{L}$ is not arbitrary

$\rightsquigarrow$ determination of all finite subgroups of $O(3)$ and $SO(3)$

Let's construct ↑

Consider G finite non-trivial subgroup of $SO(3)$, and let's identify $\{n \in \mathbb{R}^3 \mid \|n\| = 1\}$ with $\mathbb{S}^2$

We set $X = \{n \in \mathbb{S}^2 \mid Rn = n \ \exists R \in G \setminus \{\mathbb{1}\}\}$

At most $|X| \leq 2(g-1)$ with $g = |G|$

because of identity

Observe that G is a group of transformation of X.

Take $n \in X : Rn = n$. Consider $R'n$ and check $R'n \in X$.

Indeed, $\underbrace{(R'RR'^{-1})}_{\in G} R'n = R'Rn = R'n \Rightarrow R'n \in X$

$\Rightarrow X = O_1 \sqcup O_2 \sqcup \cdots \sqcup O_r$ with each O_j the orbit of a point $n \in X$.

Let S_j be the stabilizer for one point of the orbit O_j, then

$2 \leq |S_j| =: g_j \leq g$

identity + at least one R