Lemma for $su(2)_{\mathbb{C}}$ last time

$\forall d \in \mathbb{N}^* \exists !$ irreducible unitary rep. of $su(2)$ in the space of dim. d

In rep. $(h, \mathcal{V})$ $h(H)$ has eigenvalues $j, j-1, \cdots, -j$ for $j = \frac{d-1}{2}$ ⚠

Since we have, rep. of $su(3)$ of dim. $1, 3, \bar{3}, 6, \bar{6}, 8, 10, \overline{10}$

Recall that if $(h_1, \mathcal{V}_1)(h_2, \mathcal{V}_2)$ are irred. rep. of G, then

then $(h_1 \otimes h_2, \mathcal{V}_1 \otimes \mathcal{V}_2)$ is usually not an irred. rep. ⇒ can be decomposed

E.g. $3 \otimes 3 = 6 \oplus \bar{3}$; $3 \otimes \bar{3} = 8 \oplus 1$; $6 \otimes 3 = 10 \oplus 8$;

$3 \otimes 3 \otimes 3 = (6 \oplus \bar{3}) \otimes 3 = (6 \otimes 3) \oplus (\bar{3} \otimes 3) = 10 \oplus 2 \cdot 8 \oplus 1$

(unique modulo commutation and unitary equivalence)

Remark: Each semi-simple complex Lie algebra with a Cartan subalgebra of dim n_0 has n_0 indep. Casimir operators.

They are denoted by $C_2, C_3, \cdots, C_{n_0+1}$ and can be constructed with elem. of $\mathcal{L}$.

They don't $\in \mathcal{L}$ but they commute with each element of $\mathcal{L}$, it means

$[C_j, Y] = 0 \ \forall Y \in \mathcal{L}$ (one has to give a meaning to $[C_j, Y]$)

In addition, in any finite rep. of $\mathcal{L}$, one has

$h(C_j) = c_j \mathbb{1}_V$ with $c_j \in \mathbb{C}$

E.g. for $su(2)$, $C_2 = J^2 = J_1^2 + J_2^2 + J_3^2$ and $c_2 = j(j+1)$

For $su(3)$, one has 2 Casimir operators

$$C_2 := H_1^2 + H_2^2 + E_\alpha E_{-\alpha} + E_{-\alpha} E_\alpha + E_\beta E_{-\beta} + E_{-\beta} E_\beta + E_\gamma E_{-\gamma} + E_{-\gamma} E_\gamma$$

and in $D^{(\kappa_1, \kappa_2)}$ one has

$$c_2 = \frac{1}{9}(\kappa_1^2 + \kappa_1 \kappa_2 + \kappa_2^2) + \frac{1}{3}(\kappa_1 + \kappa_2)$$ ← kappa

C_3 is a polynomial of degree 3 in the generators of the algebra, (too complicated)

and in $D^{(\kappa_1, \kappa_2)}$ one has

$$c_3 = \frac{1}{9}(\kappa_1 - \kappa_2)(2\kappa_1 + \kappa_2 + 3)(\kappa_1 + 2\kappa_2 + 3)$$

IV.6 Application of SU(3) to physics

Since the elements of a Cartan subalgebra can be diagonalized simultaneously, they are often used to index families of particles.

In particular su(3) is often used. One has

$I_3 := \sqrt{3} H_1$ (isospin) and $Y := 2H_2$ (hypercharge)

People have observed that particles with similar properties gather by families of 1, 8, or 10 members, (see figures 5.19)

Such families can be generated by

$3 \otimes \bar{3} = 8 \oplus 1$ or $3 \otimes 3 \otimes 3 = 10 \oplus 2 \cdot 8 \oplus 1$

idea → Basic building block of the theory should be 3 quarks and 3 antiquarks (see figure 5.20)

We give the names u, d, s or $\bar{u}, \bar{d}, \bar{s}$ for weights of 3 and $\bar{3}$

With this idea, Figure 5.19 (a) ocorresponds to the decomposision $8 \oplus 1$ of $3 \otimes \bar{3}$ or more precisely it is called the family of MESONS of spin 0 made of 1 quark and 1 $\overline{\text{quark}}$. (see figure 5.21)

Figure 5.19 (b): BARYON DECUPLET made of 3 quarks u, d, s.

" (c): BARYON OCTET " "

Recall that a basis of $V_1 \otimes V_2$ is given by $x \otimes y$ for x a basis of V_1 and y a basis of V_2.

In picture 5.21, one uses the notation $d\bar{s}$ for $d \otimes \bar{s}$, etc.

Then for $3 \otimes \bar{3}$ the symmetric element $\frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$ corresponds to the representation 1 in $8 \oplus 1$.

Nowadays, models are much more complicated than this.

Also used $SU(5), \cdots, SO(10)$

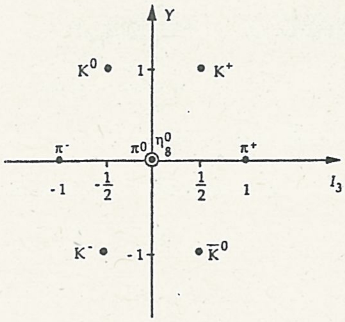


Figure 5.19(a)

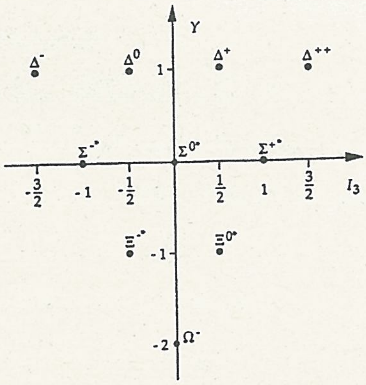


Figure 5.19(b)

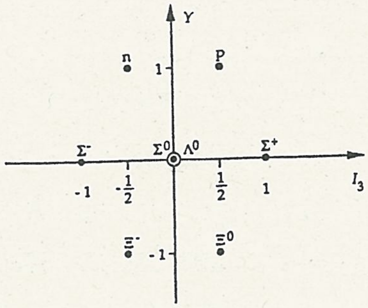
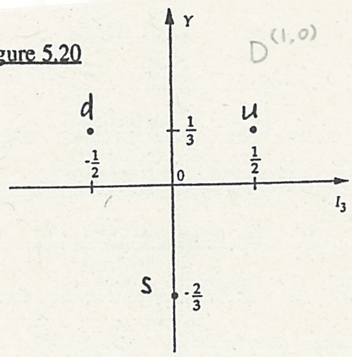


Figure 5.19(c)

Figure 5.20



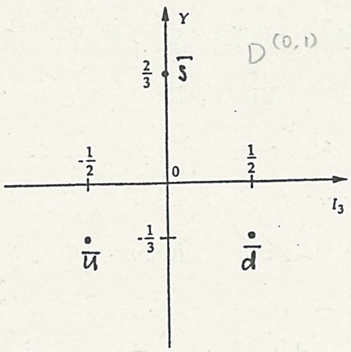
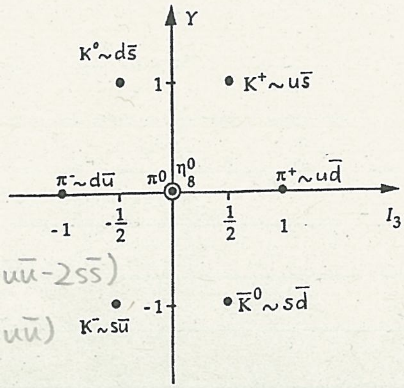


Figure 5.21



$\eta^0_8 \sim \frac{1}{\sqrt{6}}(d\bar{d} - u\bar{u} - 2s\bar{s})$

$\pi_0 \sim \frac{1}{\sqrt{2}}(d\bar{d} - u\bar{u})$

IV.7 Classification thm

Recall that semi-simple Lie algebras consist of sum of simple Lie algebras
⇒ The building blocks are simple complex Lie algebras
↳ {0} is the only proper ideal

Recall also that roots have very special properties. In the standard basis

$-2\frac{\alpha\cdot\beta}{\|\alpha\|^2} \in \mathbb{Z}$, $-2\frac{\alpha\cdot\beta}{\|\beta\|^2} \in \mathbb{Z}$ for any roots (⇒ weights) α, β

$\Leftrightarrow 2\alpha\cdot\beta = N_1\|\alpha\|^2$ and $2\alpha\cdot\beta = N_2\|\beta\|^2$ $\exists\, N_1, N_2 \in \mathbb{Z}$

$\Rightarrow \frac{\|\alpha\|}{\|\beta\|} = \sqrt{\frac{N_2}{N_1}}$ and $\frac{|\alpha\cdot\beta|^2}{\|\alpha\|^2\|\beta\|^2} = \frac{N_1N_2}{4}$; $N_1 = 0$ iff $N_2 = 0$

↳ $=: |\cos(\phi_{\alpha\beta})|^2 \in [0,1] \Rightarrow N_1N_2 \in [0,4]$

Possible (N_1, N_2):

N_2 \ N_1	0	1	2	3	4
0		×	×	×	×
1	×				
2	×			×	×
3	×		×	×	×
4	×		×	×	×

(rows 2–4: $N_1N_2 > 4$)

(N_1,N_2)	(2,2)	(1,3) ↔	(1,2) ↔	(1,4) ↔	(0,0)	(1,1)
$\phi_{\alpha,\beta}$	0° 180°	30° →150°	45° →135°	0° 180°	→90°	60° →120°

(2,2): $\|\alpha\| = \|\beta\|$ ⇒ $\alpha = \beta$ — we don't want to consider (trivial)

(1,4): $\|\alpha\| = 2\|\beta\|$ ⇒ $\alpha = 2\beta$ — impossible for simple roots

One more information: $\alpha\cdot\beta \leq 0$ (Prop. 8.11 in [Hall])
⇒ only possible angles are
90°, 120°, 135°, 150°

Then if we consider complex simple Lie algebras $\mathcal{L}$ and denote by O any simple root of $\mathcal{L}$

α β

($\|\alpha\|^2$ ↗) O O if $\phi_{\alpha\beta} = 90°$

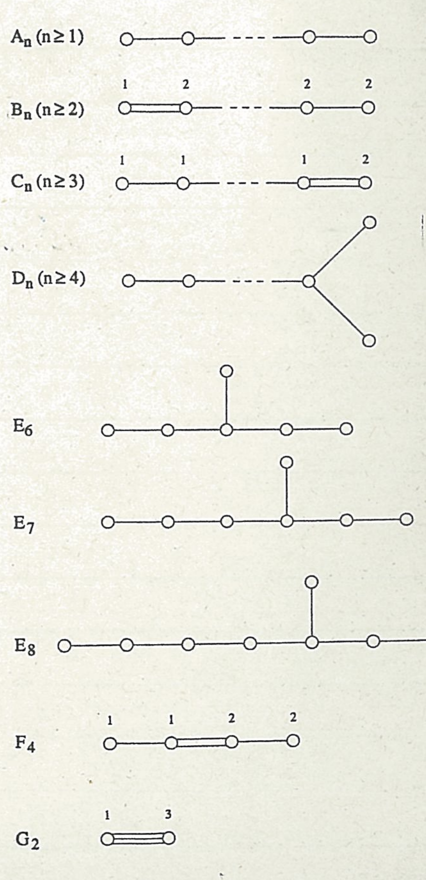
O—O if $\phi_{\alpha\beta} = 120°$

O═O if $\phi_{\alpha\beta} = 135°$

O≡O if $\phi_{\alpha\beta} = 150°$

cannot exist (see book)

By using this notation and that simple roots are linear indep, one can get the list of all simple complex Lie algebras →



	dim $\mathfrak{L}$	nombre de racines
A_n (n≥1)	n(n+2)	n(n+1)
B_n (n≥2)	n(2n+1)	$2n^2$
C_n (n≥3)	n(2n+1)	$2n^2$
D_n (n≥4)	n(2n-1)	2n(n-1)
E_6	78	72
E_7	133	126
E_8	248	240
F_4	52	48
G_2	14	12

These algebras have been extensively studied and have applications in physics and in mathematics (see Wikipedia on E_8)

Conclusion

We have seen many concepts which can be used in physics (QM) but also in mathematics. Please remember all these, and come back to the literature as often as possible.