Example for $su(2)_{\mathbb{C}}$

Let $(h,\mathcal{V})$ be a finite irreducible rep. for $su(2)_{\mathbb{C}}$, and
and let μ_{max} be the maximal weight (since $n_0 = 1$, $\mathcal{L}_0^* = \mathbb{C}$ and all $\mu \in \mathbb{R}$)
Since $\alpha = \pm\frac{1}{\sqrt{2}}$, $\Rightarrow \|\alpha\|^2 = \frac{1}{2}$ and for $\alpha = -\frac{1}{\sqrt{2}}$ one has

$$N = -2\frac{-\mu/\sqrt{2}}{1/2} = 2\sqrt{2}\mu \in \mathbb{N} \Rightarrow \mu \in \frac{\sqrt{2}}{4}\mathbb{N}$$

The other possible weights are

$$\mu, \mu - \frac{1}{\sqrt{2}}, \mu - \frac{2}{\sqrt{2}}, \cdots, \mu - \frac{1}{\sqrt{2}}(2\sqrt{2}\mu) = -\mu$$

Or equivalently if $j := \sqrt{2}\mu$ then the possible values of $\sqrt{2}$(weight) are

$$j, j-1, j-2, \cdots, -j \text{ with } 2j \in \mathbb{N}$$

The missing argument:

Lemma: for any $d \in \mathbb{N}^*$ there exists a unique irreducible representation of $su(2)$ (modulo equivalence), and in such a representation, $j = \frac{d-1}{2}$.

(This is usually in quantum mechanics)

In the setting we consider $\mathcal{L}$ with the canonical basis and let $(h,\mathcal{V})$ be an irreducible map of $\mathcal{L}$.

Let μ_{max} be the maximal weight (once $\mathbb{R}^{n_0} = (\mu_1, \cdots, \mu_{n_0})$ with $\mu_j = \mu(H_j)$ is endowed with the lexicographic order).

Clearly if α is a positive root then $\mathcal{L}_{\mu_{max}+\alpha} = \{0\}$, otherwise contradiction.

Exercise If $(h,\mathcal{V})$ is an irreducible rep. of $\mathcal{L}$, and
and μ is a weight with with $v \in \mathcal{L}_\mu$.

1) $\mathrm{span}\{v, E_\alpha v, E_\alpha E_\beta v, \cdots \mid \alpha, \beta, \cdots \in R\} = \mathcal{V}$. ($\alpha = \beta$ also considered)

2) $\mu = \mu_{max} \Rightarrow \mathrm{span}\{v, E_\alpha v, E_\alpha E_\beta v, \cdots \mid \alpha, \beta, \cdots \in R_-\} = \mathcal{V}$

$\Rightarrow \mathrm{span}\{v, E_\alpha v, E_\alpha E_\beta v, \cdots \mid \alpha, \beta, \cdots \in R_+ \text{ simple}\} = \mathcal{V}$

In addition one has

Prop: Let $(h,\mathcal{V})$ be an irreducible map of $\mathcal{L}$ and μ_{max} the maximal weight.

1) $\forall \mu$ weight: $\mu = \mu_{max} - \sum_{\substack{\alpha \in \mathbb{R}_-,\\ \alpha \text{ simple}}} n_\alpha \alpha$ with $n_\alpha \in \mathbb{N}$;

2) $\dim \mathcal{V} = \sum_{\mu \text{ weight}} \dim \mathcal{L}_\mu$;

3) $\dim \mathcal{L}_{\mu_{max}} = 1$.

Remark: In order to get all irreducible representations of $\mathcal{L}$, we
we should know all possible μ_{max}. Such formulae exist and

(IV.5)

for a given μ_{max} one has

$$\dim \mathcal{V} = \frac{\prod_{\alpha\in R_+} \alpha\cdot(\mu_{max}+\delta)}{\prod_{\alpha\in R_+}\alpha\cdot\delta} \quad \text{with } \delta = \frac{1}{2}\sum_{\alpha\in R_+}\alpha \quad \text{(Weyl Formula)}$$

and the multiplicity of each weight can be computed by the so-called Kostant's formula.

IV.5 Representations of $su(3)_{\mathbb{C}}$

Recall that $\dim(su(3)) = 8$ and $n_0 = 2$, and the matrices $H_1, H_2, E_{\pm\alpha}, E_{\pm\beta}, E_{\pm\gamma}$ are exibited in IV.3

The roots are $\alpha = (\frac{1}{2\sqrt{3}}, \frac{1}{2})$, $\beta = (-\frac{1}{2\sqrt{3}}, \frac{1}{2})$, $\gamma = (\frac{1}{\sqrt{3}}, 0)$

We set

$I_3 := \sqrt{3}\, H_1$ $\qquad I_\pm := \sqrt{6}\, E_{\pm\gamma}$ $\qquad$ (I-spin)

$U_3 := \frac{3}{2} H_2 - \frac{\sqrt{3}}{2} H_1$ $\qquad U_\pm := \sqrt{6}\, E_{\pm\beta}$ $\qquad$ (U-spin)

$V_3 := -\frac{3}{2} H_2 - \frac{\sqrt{3}}{2} H_1$ $\qquad V_\pm := \sqrt{6}\, E_{\mp\alpha}$ $\qquad$ (V-spin)

Then

$U_\pm, U_3$ leave $\{k\begin{pmatrix}1\\0\\0\end{pmatrix} \mid k\in\mathbb{C}\}$ invariant

$V_\pm, V_3$ " $\begin{pmatrix}0\\1\\0\end{pmatrix}$ "

$I_\pm, I_3$ " $\begin{pmatrix}0\\0\\1\end{pmatrix}$ "

These 3 triples generate 3 representations of $su(2)_{\mathbb{C}}$ which are not irreducible since there is an invariant subspace

Consider a $<\infty$ dim. irred. rep. $(h, \mathcal{V})$ of $su(3)_{\mathbb{C}}$, and set

$\mathcal{H}_j := h(H_j)$, $\mathcal{E}_\alpha := h(E_\alpha)$; $\mathcal{I}_3 := h(I_3)$, $\mathcal{U}_3 := h(U_3)$, $\mathcal{V}_3 := h(V_3)$

As for $su(2)$ before, $\mathcal{I}_3$, $\mathcal{U}_3$ and $\mathcal{V}_3$ are diagonalizable with evals in $\mathbb{Z}/2$ (like j before)

$\Rightarrow$ if μ is a weight, $\sqrt{3}\mu_1 = \sqrt{3}\mu(H_1) \in \mathbb{Z}/2$ and

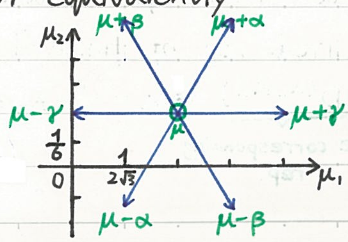
$\frac{3}{2}\mu_2 - \frac{\sqrt{3}}{2}\mu_1 \in \mathbb{Z}/2 \Leftrightarrow 3\mu_2 \in \mathbb{Z} + \sqrt{3}\mu_1 \subset \mathbb{Z}/2$

Thus $\mu_1 \in \mathbb{Z}/2\sqrt{3}$ and $\mu_2 \in \mathbb{Z}/6$; Or equivalently

and $\mu\pm\alpha = (\mu_1 \pm \frac{1}{\sqrt{3}}\frac{1}{2}, \mu_2 \pm \frac{1}{2})$

$\mu\pm\beta = (\mu_1 \mp \frac{1}{\sqrt{3}}\frac{1}{2}, \mu_2 \pm \frac{1}{2})$

$\mu\pm\gamma = (\mu_1 \pm \frac{1}{\sqrt{3}} 1, \mu_2)$



Weight diagram (for $su(3)$)

$\hat{\nu} = \mu_{max}$

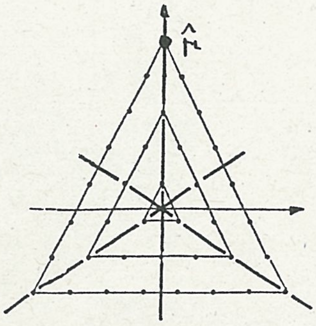


Figure 5.11(a)

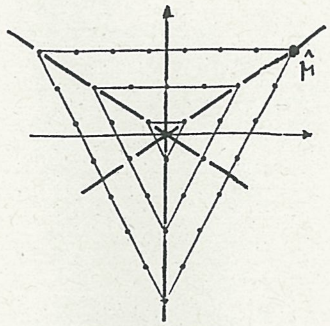


Figure 5.11(b)

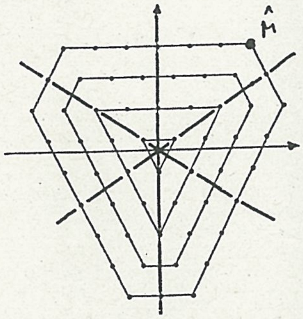


Figure 5.11(c)

If we represent the maximal weights on a so-called weight diagram, because of symmetry there are only 3 different types of position (see picture).
For simplicity we introduce the notation $(K_1, K_2) \in \mathbb{N} \times \mathbb{N}$ (?) ~~with~~

with $K_1 = 2\sqrt{3}\,\mu_{max}(H_1)$, $K_2 = 3\mu_{max}(H_2) - \sqrt{3}\,\mu_{max}(H_1)$

$\Leftrightarrow \mu_{max} = \left(\frac{K_1}{2\sqrt{3}}, \frac{K_1 + 2K_2}{6}\right)$

From any $(K_1, K_2) \in \mathbb{N} \times \mathbb{N}$ we can generate a weight diagram, and thus a irred. rep. of $su(3)_{\mathbb{C}}$

It means all irred. rep. of finite dim. are indexed by the 2 integers (K_1, K_2)

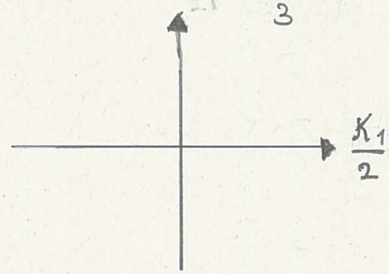
and $\dim(\underbrace{D^{(K_1, K_2)}}_{\text{the corresponding rep.}}) =: n = \frac{1}{2}(K_1 + 1)(K_2 + 1)(K_1 + K_2 + 2)$

(from Weyl formula)

$D^{(K_1, K_2)}$ with

$$\mu_{max} = \left(\frac{K_1}{2\sqrt{3}}, \frac{K_1 + 2K_2}{6} \right).$$

Representation in the basis



● = μ_{max}

⊙ = weight of multiplicity 2.

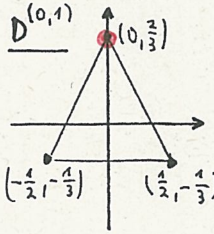


Figure 5.13(a)

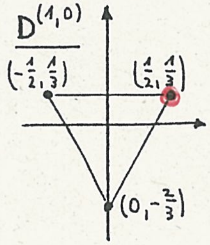


Figure 5.13(b)

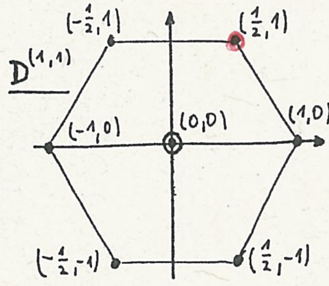


Figure 5.13(c)

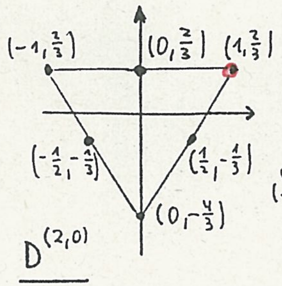


Figure 5.13(d)

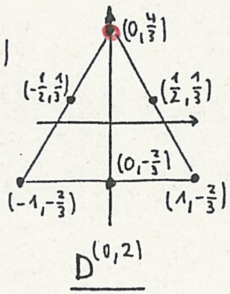


Figure 5.13(e)

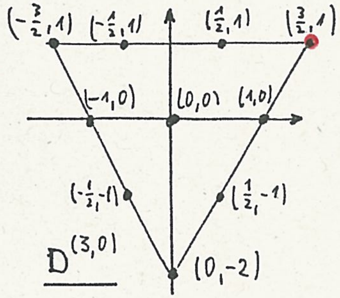


Figure 5.13(f)

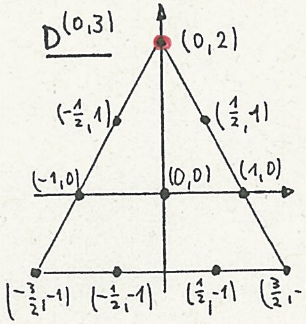


Figure 5.13(g)

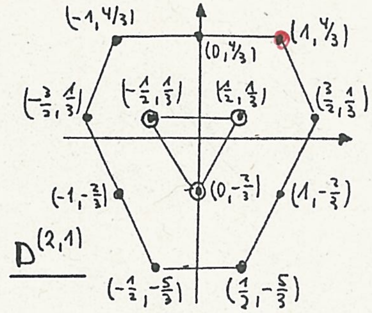


Figure 5.13(h)