In def of POSITIVE (NEGATIVE) ROOT, the "first non-zero" entry is in the reverse order: $\alpha(H_{n_0}), \alpha(H_{n_0-1}), \cdots, \alpha(H_1)$

Prop. Let $\mathcal{L}$ be a semi-simple complex Lie algebra, and consider the canonical basis $\{H_1, \cdots, H_{n_0}, E_\alpha, \cdots, E_\gamma\}$ or $\mathcal{L}$.

1) There are n_0 simple roots $\alpha^1, \cdots, \alpha^{n_0}$ ⚠ "n_0" ≠ "no"
2) These n_0 roots generate $\mathcal{L}_0^* \Rightarrow$ they are lin. indep.
3) If $\beta \in \mathcal{R}$ not simple, then $\exists a_1, \cdots, a_{n_0} \in \mathbb{Z} : \beta = a_1\alpha^1 + \cdots + a_{n_0}\alpha^{n_0}$
   with either $a_1, \cdots, a_{n_0} \geq 0$ or $a_1, \cdots, a_{n_0} \leq 0$

Recall that $su(n) = \{X \in M_n(\mathbb{C}) \mid X^* = -X$ and $tr(X) = 0\}$ and $\dim(su(n)) = n^2 - 1$ (over $\mathbb{R}$)
(→ not X^{-1}! (unitary is in $SU(n)$))

What is the dimension of any Cartan subalgebra in $su(n)_{\mathbb{C}}$?

Observation: if $X \in su(n)$ then $(iX)^* = -i(-X) = iX \Rightarrow iX$ is Hermitian

How many elements in $\{Y \in M_n(\mathbb{C}) \mid Y^* = Y$ and $tr(Y) = 0\}$ are diagonal & lin. indep?

Answer: $n - 1 =: n_0$

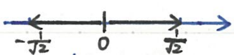
Example:

1) $su(2)$ of dim 3, and $n_0 = 1$. A basis for $su(2)_{\mathbb{C}}$ is given by the Pauli matrices.
   We choose the following basis:

$$H = \frac{1}{2\sqrt{2}}\sigma_3 = \begin{pmatrix} \frac{1}{2\sqrt{2}} & 0 \\ 0 & -\frac{1}{2\sqrt{2}} \end{pmatrix};\ \underset{(E_\alpha)}{E_+} = \frac{1}{4}(\sigma_1 + i\sigma_2) = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix};\ \underset{(E_{-\alpha})}{E_-} = \frac{1}{4}(\sigma_1 - i\sigma_2) = \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{pmatrix}$$

$\sigma_1, \sigma_2, \sigma_3$: are Pauli matrices

In this basis $\alpha = \alpha(H) = \frac{1}{\sqrt{2}}$, $ad_H(E_\alpha) = \frac{1}{\sqrt{2}}E_\alpha$, and $g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

The roots are  ; $[E_\alpha, E_{-\alpha}] = \frac{1}{\sqrt{2}}H$

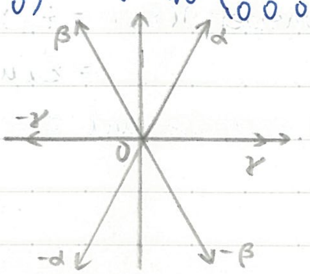
2) $su(3)$ of dim 8 and $n_0 = 2$. $\mathcal{R}$ contains 6 elements. Choose

$$H_1 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix};\ E_\alpha = \frac{1}{\sqrt{6}}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};\ E_\beta = \frac{1}{\sqrt{6}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix};\ E_\gamma = \frac{1}{\sqrt{6}}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

$$H_2 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix};\ E_{-\alpha} = \frac{1}{\sqrt{6}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix};\ E_{-\beta} = \frac{1}{\sqrt{6}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix};\ E_{-\gamma} = \frac{1}{\sqrt{6}}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In this basis g has expected form and

$$\underbrace{\underset{\hookrightarrow = \beta + \gamma}{\alpha = \left(\frac{1}{2\sqrt{3}}, \frac{1}{2}\right)};\ \underset{\hookrightarrow \text{simple}}{\beta = \left(-\frac{1}{2\sqrt{3}}, \frac{1}{2}\right)};\ \underset{\hookrightarrow \text{simple}}{\gamma = \left(\frac{1}{\sqrt{3}}, 0\right)}}_{\text{positive roots}}$$



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IV. 4 Weights of semi-simple complex Lie algebras

Let $\mathcal{L}$ be a complex Lie algebra, and let $(h, \mathcal{V})$ be a representation in $\mathcal{V}$ of finite dimension. It means $h : \mathcal{L} \mapsto L(\mathcal{V})$ s.t.

$$h(\lambda X + Y) = \lambda h(X) + h(Y) ; \; h([X,Y]) = h(X)h(Y) - h(Y)h(X) ; \; h(0) = 0.$$

Def. As for adjoint map, we look for $v \in \mathcal{V}$ s.t.

$$v \neq 0 \text{ and } h(H)v = \underbrace{\mu(H)}_{\in \mathbb{C}} v \;\; \forall H \in \mathcal{L}_0$$

In this case, v is called a WEIGHT VECTOR; and the map $\mu : \mathcal{L}_0 \mapsto \mathbb{C}$ a WEIGHT

In fact $\mu \in \mathcal{L}_0^*$ ($\Leftrightarrow \mu(H_1 + \lambda H_2) = \mu(H_1) + \lambda \mu(H_2)$)

More generally, $\forall \mu \in \mathcal{L}_0^*$ we set $\mathcal{V}_\mu = \{ v \in \mathcal{V} \mid \underbrace{\underbrace{h(\overset{\in \mathcal{L}}{H})}_{\in L(\mathcal{V})} v}_{\in \mathcal{V}} = \underbrace{\mu(H)}_{\in \mathbb{C}} v \}$ and $\dim(\mathcal{V}_\mu) =:$ MULTIPLICITY of μ

Remark: Roots are special weights when $\mathcal{V} = \mathcal{L}$. ($\Rightarrow$ multiplicity $\in \{0, 1\}$)

Choose again the canonical basis $\{H_1, \cdots, H_{n_0}, E_\alpha, \cdots, E_\gamma\}$ of $\mathcal{L}$ and set $h(H_j) =: \mathcal{H}_j$; $h(E_\alpha) =: \mathcal{E}_\alpha$; $h(H) =: \mathcal{H}$ for $H \in \mathcal{L}_0$

$$\Rightarrow \begin{cases} [\mathcal{H}, \mathcal{E}_\alpha] = \alpha(H)\mathcal{E}_\alpha ; \\ [\mathcal{H}_i, \mathcal{H}_j] = 0 ; \\ [\mathcal{E}_\alpha, \mathcal{E}_\beta] = \begin{cases} \sum_{j=1}^{n_0} \alpha(H_j)\mathcal{H}_j & \text{if } \alpha + \beta = 0 \\ J_{\alpha\beta} \mathcal{E}_{\alpha+\beta} & \text{if } \alpha + \beta \in R \\ 0 & \text{otherwise} \end{cases} \end{cases}$$

⚠ $\{\mathcal{H}_j, \mathcal{E}_\alpha\}$ are not always lin. indep.

Prop. Let $(h, \mathcal{V})$ be rep. of $\mathcal{L}$; ~~Let~~

Let μ be a weight with a weight vector $v \in \mathcal{V}_\mu \setminus \{0\}$.

1) $\mathcal{E}_\alpha v \in \mathcal{V}_{\alpha+\mu}$, and $\mu + \alpha$ is a weight if $\dim(\mathcal{V}_{\mu+\alpha}) \neq 0$;

2) The weight vectors associated with different weights are lin. indep.;

3) There exists $\leq \dim(\mathcal{V})$ weights for $(h, \mathcal{V})$.

Proof: 1) Consider $\mathcal{H}\mathcal{E}_\alpha v = \mathcal{E}_\alpha \mathcal{H} v + [\mathcal{H}, \mathcal{E}_\alpha] v$

$$= \mathcal{E}_\alpha \mu(H) v + \alpha(H)\mathcal{E}_\alpha v = \underbrace{(\mu + \alpha)}_{\in \mathcal{L}_0^*}(\mathcal{H})\mathcal{E}_\alpha v \in \mathcal{V}_{\mu+\alpha}. \quad \square$$

2) as exercise and 2) $\Rightarrow$ 3)

Def. For $\forall \alpha \in R$ and any weight $\mu \in \mathcal{L}_0^*$, we set

$$\alpha \cdot \mu := {}^T\alpha \, g \, \mu \in \mathbb{C}$$ (g: upper part of the Killing form)

Exercise:

1) $\alpha \cdot \mu$ is independent of the choice of a basis in $\mathcal{L}_0$

2) If we choose $\mathcal{V} = \mathcal{L}$, then $\|\alpha\|^2 := \alpha \cdot \alpha = \sum \alpha(H_j)^2 > 0$

Question: For which $k \in \mathbb{Z}$ one has $\mu + k\alpha$ is a weight ($\Leftrightarrow \mathcal{V}_{\mu + k\alpha} \neq \{0\}$) ?

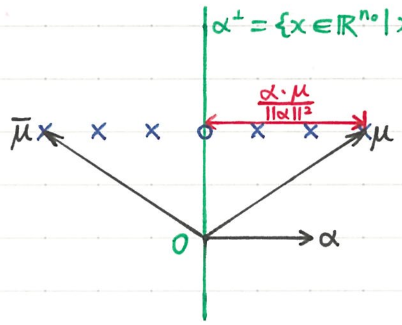
Lemma (technical but based only on $[H_j, E_\alpha] = \cdots$ and $[E_\alpha, E_\beta] = \cdots$)

1) $\alpha \cdot \mu \in \mathbb{R}$

2) $N := -2 \frac{\alpha \cdot \mu}{\|\alpha\|^2} \in \mathbb{Z}$ (can be positive or negative)

3) $\forall k \in \mathbb{Z} \cap [0, N]$: $\mu + k\alpha$ is a weight $\longleftarrow$ in this case replace $[0,N]$ by $[N,0]$

4) In the standard basis with $\alpha_j := \alpha(H_j) \in \mathbb{R}$, we have $\mu_j := \mu(H_j) \in \mathbb{R}$ ($\Rightarrow$ (1))



$\bar{\mu} := \mu - 2 \frac{\alpha \cdot \mu}{\|\alpha\|^2}$

$\times$ are weights ; $\circ$ is sometimes a weight