IV.3

Thm. $\mathcal{L}$ is semi-simple iff $\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \cdots \oplus \mathcal{L}_N$ with each $\mathcal{L}_j$ simple Lie algebra

Example: For $\mathcal{L}_1 \oplus \mathcal{L}_2$ They don't speak to each other

$[X_1 + X_2, Y_1 + Y_2] := [X_1, Y_1] + [X_2, Y_2]$

Corollory: Any semi-simple Lie algebra of dim 2 or 3 is simple.

(Since any simple Lie algebra is of dim >1)

IV.3 Roots of semi-simple complex Lie algebra

Recall that any real Lie algebra can be complexified.

Recall that for any $X \in \mathcal{L}$, $ad_X : \mathcal{L} \mapsto \mathcal{L}$ linear

$\Rightarrow$ We can look at eigenvalues of ad_X, which means

$\lambda \in \mathbb{C} : [X, Y] = ad_X(Y) = \lambda Y \; \exists Y \in \mathcal{L}$

If $\mathcal{L}$ is of dim n, then $\det(ad_X - \lambda \mathbb{1})$ is a polynomial with deg n, with n roots

Clearly 0 is an eigenvalue since $ad_X(X) = [X, X] = 0 = 0X$

Recall also that ad_X admits n generalized eigenvectors

(Jordan normal form of a matrix)

Def. Let $\mathcal{L}$ be a semi-simple complex Lie algebra.

A CARTAN SUBALGEBRA $\mathcal{L}_0$ of $\mathcal{L}$ is a maximal abelian subalgebra of $\mathcal{L}$ with all ad_X {with $X \in \mathcal{L}_0$} simultaneously diagonalizable.

It means $\mathcal{L}_0$ is a complex vector space s.t.

1) $\forall X_1, X_2 \in \mathcal{L}_0 : [X_1, X_2] = 0$

2) If for $Y \in \mathcal{L}$, $[X, Y] = 0 \; \forall X \in \mathcal{L}_0$, then $Y \in \mathcal{L}_0$

3) $\forall X \in \mathcal{L}_0$: ad_X is diagonalizable

Remark: One should show that for semi-simple Lie algebras, such Cartan subalgebra always exists;

and if there are >1, then they have the same dimension.

We call the RANK of $\mathcal{L} =: n_0 < n$ the dim of $\mathcal{L}_0$.

$\hookrightarrow$ because of semi-simple

Let us fix $\mathcal{L}_0$ a Cartan subalgebra in $\mathcal{L}$, $\quad$ ↱ for ad_X is diagonalizable ③

and choose a basis $\{Y_1, \cdots, Y_n\}$ of $\mathcal{L}$ s.t. $ad_X(Y_j) = \lambda_j(X) Y_j \ \forall X \in \mathcal{L}_0$.

Let us observe that if $X, X' \in \mathcal{L}_0$ and $\alpha \in \mathbb{C}$ then $X + \alpha X' \in \mathcal{L}_0$, so

$$\lambda_j(X+\alpha X')Y_j = ad_{X+\alpha X'}(Y_j) = [X+\alpha X', Y_j] = [X, Y_j] + \alpha [X', Y_j]$$
$$= ad_X(Y_j) + \alpha\, ad_{X'}(Y_j) = (\lambda_j(X) + \alpha \lambda_j(X'))Y_j$$

$\Rightarrow \lambda_j(X+\alpha X') = \lambda_j(X) + \alpha \lambda_j(X') \Rightarrow \lambda_j : \mathcal{L} \mapsto \mathbb{C}$ is linear $\forall j = 1, \cdots, n$

$:\Leftrightarrow \lambda_j \in \mathcal{L}_0^*$ (DUAL of $\mathcal{L}_0$) (of dim n_0)

Remark: Since $ad_X(Y) = 0$ if $X, Y \in \mathcal{L}_0$, we can choose a basis of $\mathcal{L}$ s.t.

$$\{\underbrace{Y_1, \cdots, Y_{n_0}}_{\in \mathcal{L}_0}, Y_{n_0+1}, \cdots, Y_n\}$$

Remark: Observe that $\lambda_j(X) = 0$ if $j \in \{1, \cdots, n_0\}$ and $X \in \mathcal{L}_0$

Is it possible that $\lambda_j(X) = 0 \ \forall j \in \{1, \cdots, n\}$ and $X \in \mathcal{L}_0$?

No. Because any $Y \in \mathcal{L}_0^\perp$ cannot commute with all elements of $\mathcal{L}_0$ (maximality assumption) ②

Def. (independent of the choice of a basis)

A ROOT of $\mathcal{L}$ (relative to a fixed Cartan subalgebra $\mathcal{L}_0$) is an element $\alpha \in \mathcal{L}_0^*$ ($\Leftrightarrow \alpha : \mathcal{L}_0 \mapsto \mathbb{C}$), $\alpha \neq 0$ s.t. $\exists Y \in \mathcal{L} \setminus \{0\} : ad_X(Y) = \alpha(X) Y \ \forall X \in \mathcal{L}_0$.

The set of all roots is denoted by $R \subset \mathcal{L}_0^*$

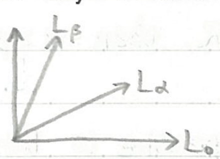
It corresponds to the generalization of an eigenvalue.

For any $\alpha \in R$ we set

$$\mathcal{L}_\alpha := \{Y \in \mathcal{L} \mid ad_X(Y) = \alpha(X) Y \ \forall X \in \mathcal{L}_0\} \neq \mathcal{L}_0 \text{ (or } \alpha = 0)$$

$\Rightarrow$ Since all ad_X can be diagonalized simultaneously, one infers that

$$\mathcal{L} = \mathcal{L}_0 \oplus \left(\bigoplus_{\alpha \in R} \mathcal{L}_\alpha \right)$$



⚠ No notion of orthogonality

In this representation $ad_X = 0 \oplus \bigoplus_{\alpha \in R} \alpha(X)$

Exercise: think about this.

We now generalize $\mathcal{L}_\alpha = \begin{cases} \mathcal{L}_\alpha & \text{if } \alpha \in R \\ \{0\} & \text{if } \alpha \notin R \text{ but } \alpha \in \mathcal{L}_0^* \end{cases}$

Lemma: $\forall \alpha, \beta \in \mathcal{L}_0^*$, $X_\alpha \in \mathcal{L}_\alpha$, $X_\beta \in \mathcal{L}_\beta$:

$[X_\alpha, X_\beta] \in \mathcal{L}_{\alpha+\beta}$ $\quad$ Proof as exercise, use Jacobian identity. Do the several cases separately. ~~and~~

$\Rightarrow$ If $\alpha, \beta \in R$ but $\alpha + \beta \notin R$ then $[X_\alpha, X_\beta] = 0$; but if $\alpha + \beta = 0$ then $[X_\alpha, X_\beta] \in \mathcal{L}_0$

Prop. 1) If $\alpha \in R$ then $-\alpha \in R$
2) $\dim \mathcal{L}_\alpha = 1$
3) $\mathrm{span}(\alpha \mid \alpha \in R) = \mathcal{L}_0^*$

} Book of Hall, P165~166

With an additional change of basis, one can construct a basis
$\{H_1, \cdots, H_{n_0}, E_\alpha, \cdots, E_{\mu}\}$ of $\mathcal{L}$ s.t.

1) $[H_j, H_k] = 0$
2) $[H_j, E_\alpha] = \alpha(H_j) E_\alpha$ with $\alpha(H_j) \in \mathbb{R}$
3) $[E_\alpha, E_\beta] = \begin{cases} \sum_{j=1}^{n_0} \alpha(H_j) H_j & \text{if } \alpha+\beta = 0 \\ J_{\alpha\beta} E_{\alpha+\beta} & \text{if } \alpha+\beta \in R \\ 0 & \text{otherwise} \end{cases}$

In this basis $(g_{ij})_{ij} = (\mathrm{tr}(ad_{x_i} ad_{x_j}))_{ij} =$

$$\begin{pmatrix} \overbrace{\begin{matrix}1 & & 0\\ & 1 & \\ 0 & & 1\end{matrix}}^{n_0} & & & & 0 \\ & \begin{matrix}0&1\\1&0\end{matrix} & & & \\ & & \begin{matrix}0&1\\1&0\end{matrix} & & \\ & & & \ddots & \\ 0 & & & & \begin{matrix}0&1\\1&0\end{matrix}\end{pmatrix} \in M_n(\mathbb{R})$$

Def. Since $\alpha(H_j) \in \mathbb{R}$ in this basis, we say that the root
for $j \in \{1, \cdots, n_0\}$
α is POSITIVE ($:\Leftrightarrow \alpha \in R_+$) if the first non-zero entry of $\alpha(H_j)$ is positive
α is NEGATIVE ($:\Leftrightarrow \alpha \in R_-$) if 〃 negative

$\Rightarrow R = R_+ \sqcup R_-$

If $\alpha, \beta \in R$, then $\alpha > \beta :\Leftrightarrow \alpha - \beta \in R_+$
↳ called the LEXICOGRAPHIC ORDER on $\mathbb{R}^n$

Def. With respective to this basis a root is SIMPLE
if it cannot be expressed as a linear combination of other positive roots.