No. 11

Date 2018 6 20

IV Semisimple theory

$\rightsquigarrow SU(n)$

IV.1) Complexification and linear independence

Note: $SU(n)$ is a Lie group and $su(n)$ is a Lie algebra.

Recall that a basis of $su(2)$ is given by

$$\left\{\frac{1}{2}\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}, \frac{1}{2}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -i\end{pmatrix}\right\}$$ They are lin. indep. on $\mathbb{R}$ but also on $\mathbb{C}$

On the other hand, a basis for $sl(2,\mathbb{C})$ is

$$\left\{\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}, \begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}, \begin{pmatrix}0 & i\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ i & 0\end{pmatrix}\right\}$$

and they are lin. indep. on $\mathbb{R}$ but not on $\mathbb{C}$.

Let us define the complexification of any real Lie algebra $\mathcal{L}$ of dim n.

Consider $\mathcal{L}\oplus\mathcal{L}$ with

$$(\lambda+i\mu)\overset{\text{"}X+iY\text{"}}{(X,Y)} := (\lambda X-\mu Y, \mu X+\lambda Y) \quad \forall \lambda,\mu\in\mathbb{R}, X,Y\in\mathcal{L}.$$

⚠ Exercise: Check that this defines a complex vector space of dim n with $(X_1,0),\cdots,(X_n,0)$ with $\{X_1,\cdots,X_n\}$ a basis of $\mathcal{L}$.

$$[(X,Y),(X',Y')] := ([X,X']-[Y,Y'], [X,Y']+[Y,X'])$$

Lemma: $\mathcal{L}\oplus\mathcal{L}$ with the above scalar mult. & the above $[\,,\,]$ is a Lie algebra over $\mathbb{C}$ of dim n.

Proof as exercise

Def. This $\mathcal{L}\oplus\mathcal{L}$ is called the COMPLEXIFICATION of $\mathcal{L}$ and denoted by $\mathcal{L}_{\mathbb{C}}$.

Lemma: Iff $\mathcal{L}$ has a basis also lin. indep. over $\mathbb{C}$ and is a real linear Lie algebra then the map

$$\phi: \mathcal{L}_{\mathbb{C}} \mapsto M_n(\mathbb{C}),\ \phi(X,Y) = X+iY$$

defines an injective homomorphism, and it is an isomorphism on its image on $M_n(\mathbb{C})$.

Proof as exercise

Exercise

$$gl(n,\mathbb{R})_{\mathbb{C}} \simeq gl(n,\mathbb{C}) \qquad su(n)_{\mathbb{C}} \simeq sl(n,\mathbb{C})$$
$$u(n)_{\mathbb{C}} \simeq gl(n,\mathbb{C}) \qquad sl(n,\mathbb{R})_{\mathbb{C}} \simeq sl(n,\mathbb{C})$$

IV. 2

Exercise

1) $X \in su(n) \Leftrightarrow X^* = -X$ and $tr(X) = 0$ (us exponensial)

2) $su(n)$ is of real dimension $n^2 - 1$

3) Any basis of $su(n)$ is lin. indep. over $\mathbb{C}$

$\rightsquigarrow$ the second lemma applies

IV. 2) Properties of Lie algebra

Def. 1) a SUBALGEBRA of $\underbrace{\text{a Lie algebra}}^{\text{of } \mathcal{L}}$ is a subspace $\mathcal{L}^1$ & s.t.

$[X, Y] \in \mathcal{L}^1 \Leftarrow X, Y \in \mathcal{L}^1$

2) A subspace $\mathcal{L}'$ of $\mathcal{L}$ is an IDEAL if

$[X, Y] \in \mathcal{L}' \ \forall X \in \mathcal{L}'$ and $Y \in \mathcal{L}$

It is a PROPER IDEAL if $\mathcal{L}' \neq \mathcal{L}$

3) The CENTER of $\mathcal{L}$ is defined by

$\{X \in \mathcal{L} \mid [X, Y] = 0 \ \forall Y \in \mathcal{L}\}$

Note that the center is always an abelian ideal.

Def. A Lie algebra $\mathcal{L}$ with $\dim \mathcal{L} > 1$ is SIMPLE if

$\{0\}$ is the only proper ideal of $\mathcal{L}$.

And $\mathcal{L}$ with $\dim \mathcal{L} > 1$ is SEMI-SIMPLE if

$\{0\}$ is the only abelian proper ideal of $\mathcal{L}$.

Def. A connected Lie group is SIMPLE if

it does not contain any proper normal Lie subgroup. $\mathbb{R} \triangleright \mathbb{Z}$ but $\mathbb{Z}$ is not a Lie gro

$\mathbb{R}$ is still simple

A connected Lie group is SEMI-SIMPLE if

it deos not contain any normal abelian proper Lie subgroup.

Lemma. Let G be a $\underbrace{\text{connected}}_{\text{linear}}$ Lie group and $\mathcal{L}(G)$ its Lie algebra (over $\mathbb{R}$)

1) A Lie subgroup G' is normal iff $\mathcal{L}(G')$ is an ideal in $\mathcal{L}(G)$

2) G is simple iff $\mathcal{L}(G)$ is simple

3) G is semi-simple iff $\mathcal{L}(G)$ is semi-simple

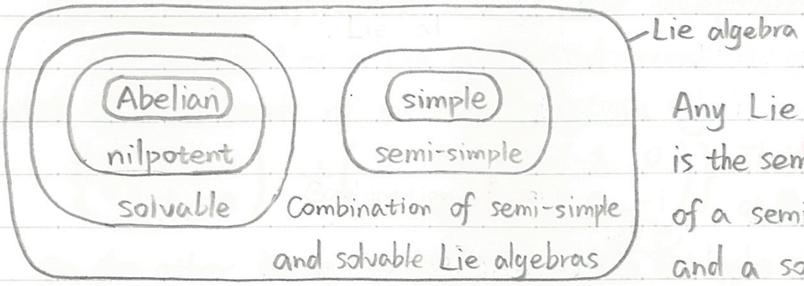
} small exercises

4) $\mathcal{L}(G)_{\mathbb{C}}$ is semi-simple iff ↑

5) If $\mathcal{L}(G)_{\mathbb{C}}$ is simple then $\mathcal{L}(G)$ is simple

} more deep

Remark (task for mathematitians)



Any Lie algebra is the semi-direct product of a semi-simple Lie algebra and a solvable Lie algebra

Let $\mathcal{L}$ be a Lie algebra and consider its adjoint representation defined by

$$\begin{aligned} ad : \mathcal{L} &\mapsto L(\mathcal{L}) \\ X &\mapsto ad_X \quad \text{with } ad_X(Y) = [X,Y] \end{aligned}$$

linear operators on the vector space $\mathcal{L}$

indeed $ad_X \in L(\mathcal{L})$ since

$$ad_X(Y+\lambda Z) = [X, Y+\lambda Z] = [X,Y] + \lambda [X,Z]$$

It is a representation since

- $ad_0 = 0$
- $ad_{X+\alpha Y} = ad_X + \alpha\, ad_Y$ (since [,] is bi-linear) (*)
- $ad_{[X,Y]} = ad_X ad_Y - ad_Y ad_X$ (use Jacobian identity)

Note that

$$ad_X([Y,Z]) = [ad_X(Y), Z] + [Y, ad_X(Z)]$$ because of Jacobian identity

Def. the KILLING FORM of $\mathcal{L}$ is the symmetric bi-linear map $\Leftarrow(*)$

$$K : \mathcal{L}\times\mathcal{L} \mapsto \mathbb{C}, \quad K(X,Y) = tr(ad_X\, ad_Y) \in \mathbb{C}$$

Exercise: 0) Illustrate the theory in this page with $\mathcal{L} = su(2)$

1) If $\{Y_1, \cdots, Y_n\}$ is a basis of $\mathcal{L}$ with
$$[Y_i, Y_j] = \sum_{k=1}^{n} c_{ij}^k Y_k$$
Then $g_{ij} := K(Y_i, Y_j) = \sum_{k,r} c_{ir}^k c_{jk}^r$

2) $K([X,Y], Z) = K(X, [Y,Z]) \; \forall X, Y, Z \in \mathcal{L}$

3) $\left[K(X,Y) = 0 \;\forall Y \in \mathcal{L} \Rightarrow X = 0\right] \Leftrightarrow \det\left((g_{ij})_{i,j=1}^{n}\right) \neq 0$

In such a case we say that the Killing form is non degenerate

A quite important thm (Cartan's criterion) :

A Lie algebra $\mathcal{L}$ is semi-simple iff its Killing form is non-degenerate.

Lemma: A semi-simple connected Lie group is compact iff the Killing form of its Lie algebra is negative definite. It means

$$K(X,X) < 0 \quad \forall X \in \mathcal{L}, X \neq 0$$

Example: $su(n)$, one has $K(X,Y) = 2n \cdot tr(XY)$ and it follows that

$$K(X,X) = 2n \cdot tr(XX) = -2n \cdot tr(X^* X) \#$$

$$= -2n \sum_{j=1}^{n} \langle e_j, X^* X e_j \rangle = -2n \sum_{j=1}^{n} \langle X e_j ; X e_j \rangle = -2n \sum_{j=1}^{n} \|X e_j\|^2 < 0$$

By knowing that $SU(n)$ is semi-simple, it follows that it is compact.