In compact Lie groups, like finite groups, all representations are equivelant to some unitary groups; but if the Lie group is not compact, not true.
So we consider not only unitary valued but more general representations with values in $GL(V)$

example : $(\mathbb{R},+) \ni x \mapsto e^x \in M_1(\mathbb{R})$

In III,2 last time,

$$Y_j := \lim_{\varepsilon\to 0} \frac{\varphi_0^{-1}(\varepsilon E_j) - \mathbb{1}}{\varepsilon} \in M_n(\mathbb{C})$$

Let $\mathbb{K}$ be either $\mathbb{C}$ or $\mathbb{R}$.

Def. A LIE ALGEBRA $\mathcal{L}$ over $\mathbb{K}$ is a (finite dim.) vector space over $\mathbb{K}$ with a composition $[\cdot,\cdot] : \mathcal{L}\times\mathcal{L} \mapsto \mathcal{L}$ s.t. $\forall X, Y, Z \in \mathcal{L}, \alpha, \beta \in \mathbb{K}$:

1) $[\alpha X + \beta Y, Z] = \alpha[X,Z] + \beta[Y,Z]$
2) $[X,Y] = -[Y,X]$
3) $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$ (Jacobian identity)

Remark: if $X, Y, Z \in M_n(\mathbb{C})$ and $[X,Y] := XY - YX$ (COMMUTATOR) then the 3 conditions are satisfied.
So we only have to check that
$X, Y \in \mathcal{L} \Rightarrow [X,Y] \in \mathcal{L}$

Def. Given a basis $\{Y_1, \cdots, Y_n\}$ of $\mathcal{L}$, the coefficients $\{c_{ij}^k\}_{i,j,k=1}^n$ defiend by $[Y_i, Y_j] = \sum_{k=1}^n c_{ij}^k Y_k$ are called STRUCTURE CONSTANTS or STRUCTURE COEFFICIENTS of $\mathcal{L}$.
$\Rightarrow c_{ij}^k = -c_{ji}^k$ and relation from Jacobian identity.

Lemma: Let G be a linear Lie group and $\mathcal{L}(G)$ be its tangent space. Then $\mathcal{L}(G)$ is a Lie algebra of the same dimension. (Proof as exercise)

Sketch of the proof: One just has to show $[Y_i, Y_j] \in \mathcal{L}(G)$.
Consider smooth curves
$(-1,1) \ni t \mapsto A(t) \in G$ with $A(0) = \mathbb{1}$ and $A'(0) = Y_i$
$(-1,1) \ni t \mapsto B(t) \in G$ with $B(0) = \mathbb{1}$ and $B'(0) = Y_j$
Consider $(-1,1) \ni t \mapsto A(\sqrt{t})B(\sqrt{t})A(\sqrt{t})^{-1}B(\sqrt{t})^{-1}$ (*)
Then $A(\sqrt{t}) = \mathbb{1} + \sqrt{t}Y_i + \cdots$, $B(\sqrt{t}) = \mathbb{1} + \sqrt{t}Y_j + \cdots$, $A(\sqrt{t})^{-1} = \mathbb{1} - \sqrt{t}Y_i + \cdots$, $B(\sqrt{t})^{-1} = \mathbb{1} - \sqrt{t}Y_j + \cdots$
Then $(*) = \mathbb{1} + t[Y_i, Y_j] + \cdots \Rightarrow (*)'(0) = [Y_i, Y_j]$ □

Important properties of G or $\mathcal{L}(G)$:

Some proofs are very nice but too long.

Let G be a Lie group and $\mathcal{L}(G)$ be its Lie algebra.

1) $\forall X \in \mathcal{L}(G), t \in \mathbb{R}: \exp(tX) := \sum_{j=0}^{\infty} \frac{1}{j!}(tX)^j \in G$

2) $\exp(sX)\exp(tX) \overset{\text{multiplication in } G}{=} \exp((s+t)X)$; $\exp(tX)^{-1}_{\text{in } G} = \exp(-tX)$

3) $t \mapsto \exp(tX)$ is the only one-parameter subgroup of G satisfying

$\frac{d}{dt}\exp(tX)|_{t=0} = X$.

Prop. (Same framework)

a) $\exists$ an open set $\mathcal{V} \subset G$ containing $\mathbb{1}$ s.t.

1) $\forall A \in \mathcal{V}\ \exists X \in \mathcal{L}(G): A = \exp(X)$

2) $\forall A \in \mathcal{V}\ \exists B \in \mathcal{V}: A = B^2 := BB$

b) $\forall A \in G_0\ \exists X_1, \cdots, X_N \in \mathcal{L}(G): A = \exp(X_1)\exp(X_2)\cdots\exp(X_n)$ (not always unique)

c) If G is compact, we can choose $N = 1$ ($\Leftrightarrow A = \exp(X_1)$)

⚠ $\exp(X_1)\cdots\exp(X_n) \neq \exp(X_1 + \cdots + X_n)$ in general

In fact $\exp(X)\exp(Y) = \exp(f(X,Y))$

with $f(X,Y)$ called the CAMPBELL-BAKER-HAUSDORFF FORMULA. ($\in \mathcal{L}(G)$)

Exercise: find $f(X,Y)$. (a series)

What about the relation between representations of G and ones of $\mathcal{L}(G)$?

Def: a REPRESENTASION of a Lie algebra $\mathcal{L}$ is

a pair (h, V) with V a vector space and $h: \mathcal{L} \mapsto \underset{\text{linear map}}{L(V)}$ a homomorphism

It means $\begin{cases} h(X+\alpha Y) = h(X) + \alpha h(Y) \\ h([X,Y]) = h(X)h(Y) - h(Y)h(X) \\ h(\mathbf{0}) = 0 \end{cases}$ $\begin{pmatrix} \mathbb{1}f = f\ \forall f \in V \\ 0f = 0\ \forall f \in V \end{pmatrix}$

Lemma: Let (U, V) be a representation of a Lie group G

in a finite dimensional vector space V.

Then $\Gamma: \mathcal{L}(G) \mapsto L(V)$, $\Gamma(X) := \frac{d}{ds} U(\exp(SX))|_{s=0}$

defines a representation of $\mathcal{L}(G)$.

In addition $\underbrace{\exp(\underbrace{S\Gamma(X)}_{\in L(V)})}_{\in GL(V)} = \underbrace{U(\exp(SX))}_{\in GL(V)}$ (*

Proof as exercise

⚠ A kind of converse is not true:
a representation of a Lie algebra does not define
a representation of a unique Lie group by (*).
If two Lie groups are isomorphic close to the identity,
then the corresponding Lie algebras are isomorphic.

Application

If $\{e^{-itH}\}_{t\in\mathbb{R}}$ with $e^{-itH}\in\mathcal{B}(\mathcal{H})$ describes the evolution of a quantum system

(H: self-adjoint operator (extension of Hermitian matrix into ∞-dim Hilbert spaces))

And if $\exists$ a Lie group G and unitary rep. $(\mathcal{U},\mathcal{H})$ s.t.

$$U(a)e^{-itH}=e^{-itH}U(a)\ \forall a\in G$$

Then any $X\in\mathcal{L}(G)$ defines a constant of motion.

More precisely $\Gamma(X):\mathcal{H}\mapsto\mathcal{H}$ which satisfies (not necessarily $\Gamma(X)\in\mathcal{B}(\mathcal{H})$)

$$e^{-itH}\Gamma(X)=\Gamma(X)e^{-itH}\Leftrightarrow e^{-itH}\Gamma(X)e^{itH}=\Gamma(X)$$

III.3 SO(3), O(3) and SU(2)

Recall that $\exists\phi: SU(2)\mapsto SO(3)$ surjective and with kernal $\{\mathbb{1},-\mathbb{1}\}$

Prop. (proofs as exercises)

1) O(3), SO(3) and SU(2) are $\underbrace{\text{compact}}_{\text{maybe difficult to show}}$ Lie groups.
2) O(3) is not connected.
3) SO(3) is connected but not simply connected.
4) SU(2) is simply connected.
5) SO(3) and SU(2) isomorphic $\underbrace{\text{near the identity}}_{\text{the neighborhood}}$ (see the note for metric & topological spaces in the Tue seminar last term) $(\mathcal{L}(SO(3))\simeq\mathcal{L}(SU(2)))$
6) The Pauli matrices $\sigma_1,\sigma_2,\sigma_3$ generate SU(2) in the following sense:

$$\{\underbrace{-\tfrac{i}{2}\sigma_1}_{Y_1},\ \underbrace{-\tfrac{i}{2}\sigma_2}_{Y_2},\ \underbrace{-\tfrac{i}{2}\sigma_3}_{Y_3}\}\text{ generate } SU(2) \text{ with } [Y_i,Y_j]=\sum_k\varepsilon_{ijk}Y_k \qquad (*)$$

$$\text{with } \cancel{\varepsilon_{ijk}}=\begin{cases}\varepsilon_{123}=\varepsilon_{231}=\varepsilon_{312}=1\\ \varepsilon_{132}=\varepsilon_{213}=\varepsilon_{321}=-1\\ \varepsilon_{ijk}=0 \text{ otherwise}\end{cases}$$

7) The following 3 matrices define the Lie algebra of SO(3)

$$Y_1=\begin{pmatrix}0&0&0\\0&0&-1\\0&1&0\end{pmatrix},\ Y_2=\begin{pmatrix}0&0&1\\0&0&0\\-1&0&0\end{pmatrix},\ Y_3=\begin{pmatrix}0&-1&0\\1&0&0\\0&0&0\end{pmatrix}$$

with the same relations of (*)