Groups & Representations

I) Groups

I.1) Basic def

Def. A group is a set G together with
a map $G \times G \to G$ (denoted by "$\cdot$", " ", or "$+$") && satisfying:
$\forall\, a,b,c \in G$

1) $(ab)c = a(bc)$ ASSOCIATIVITY
2) $\exists e \in G : ea = ae = a$ IDENTITY ELEMENT
3) $\forall a \in G\ \exists a^{-1} \in G : aa^{-1} = e$ EXSISTANCE of INVERSE

⚠ If use "$+$" notation, $a^{-1} =: -a$, $e =: 0$; for "$\cdot$" notations $e =: 1$

Remark

1) $e^{-1} = e$, $a^{-1}a = e$, $(a^{-1})^{-1} = a$, $(ab)^{-1} = b^{-1}a^{-1}$, a^{-1}, e are unique.
2) If $ab = ac$ then $b = c$; If $ba = ca$ then $b = c$

Def.

G is Abelian (or commutative) if $\forall a,b \in G : ab = ba$

G is finite if containing only a finite number of elements, ($\Leftrightarrow$: $\overbrace{|G|}^{\text{number of elements of } G} < \infty$)

Examples

1) $(\mathbb{Z}, +)$ $(\mathbb{R}, +)$ $(\mathbb{R}_+, \cdot)$ $\mathbb{R}_+ := (0, +\infty)$
2) Cyclic group C_n with $C_n = \{e, a, a^2 = aa, a^3, a^4, \cdots, a^{n-1}\}$ $(e \equiv a^0 \equiv a^n)$
   with $a^j a^k = a^{j+k \bmod n}$, $(a^j)^{-1} = a^{n-j}$ (Abelian group)
3) Symmetric group S_n = group of permutations of n elements
   - It contains $n!$ elements
   - Not Abelian (if $n \geq 3$)

   For example, $n = 3$, the elements are:

$$\underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}}_{=e} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

with
$$\begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ \underline{1} & \underline{3} & \underline{2} \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ \underline{2} & \underline{3} & \underline{1} \end{pmatrix}$$

4) $GL(n,\mathbb{R}), GL(n,\mathbb{C})$: $n\times n$ invertible matrices with multiplication

$SL(n,\mathbb{R}), SL(n,\mathbb{C})$: invertible matrices with $\det = 1$

$U(n)$: inv. matrices s.t. $U^* = U^{-1}$ $(U^* := \overline{U^T})$ →complex conjugate $(\Rightarrow |\det U| = 1)$

$SU(n)$: $\{M \in U(n) \mid \det M = 1\}$

$O(n) \subset GL(n,\mathbb{R})$ with $A^t = A^{-1}$ $(\Rightarrow \det A = \pm 1)$

$SO(n) \subset O(n)$ with $\det(A) = 1$

Def. A subgroup G_0 is a subset of group G which (G_0) is a group.

G_0 is proper if $G_0 \neq G$ and non-trivial ~~is~~ if $G_0 \neq \{e\}$

Check in examples: which are subgroups? Does C_n contain subgroups?

Def. For $a, b \in G$, a is conjugate to b if

$\exists c \in G : a = cbc^{-1}$

Then we white $a \sim b$ when a is conjugate to b.

Remark: $\sim$ is an equivalence relation, indeed:

1) $a \sim a$ (choose $c = e$) REFLEXIVITY

2) $b \sim a$ $(a = cbc^{-1} \Leftrightarrow c^{-1}ac = b)$ SYMMETRY

3) $b \sim d \Rightarrow a \sim d$ (check it) TRANSITIVITY $\left.\begin{array}{l} a = mbm^{-1} \\ b = ndn^{-1} \end{array}\right\} \Rightarrow \begin{array}{l} a = mndn^{-1}m^{-1} \\ = mnd(mn)^{-1} \end{array}$

Def. a, b are in the same equivalence class (or conjugacy class) if $a \sim b$.

Remark:

- Each element $a \in G$ is in a single conjugacy class.
- e generates a class on its own.
- If G is Abelian, each element generates its own class.

More generaly, let G_0 be subgroup of G

and set $cG_0c^{-1} := \{cac^{-1} \mid a \in G_0\}$, $(\forall a, b \in G: cac^{-1}cbc^{-1} = cabc^{-1} \in G_0)$

Then cG_0c^{-1} is also a subgroup of G

Def: cG_0c^{-1} is a subgroup conjugated to G_0

- If $\forall c \in G : cG_0c^{-1} = G_0$ then G_0 is called normal or invariant.

  (written $G_0 \triangleleft G$)

No.

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Examples:

~~Ex~~. · Consider $(\mathbb{R},+) = G$, and $(\mathbb{Z},+) = G_0$ is a normal subgroup.

· $G = GL(n,\mathbb{C})$, $G_0 = \mathbb{C}^* \mathbb{1}_{n\times n}$ is normal (and Abelian)

↳ not necessary for normal subgroup

Def. The center $Z(G)$ of a group G is defined by

$\{a\in G \mid ab = ba \ \forall b\in G\}$

Exercise: $Z(G)$ is an Abelian & normal subgroup of G

Def. · G is simple if $\{e\}$ is the only proper normal subgroup of G.

· G is semi-simple if $\{e\}$ is the only proper normal Abelian subgroup of G.

Def. Let G_0 be a subgroup of G, and $a,b\in G$

We set $a \overset{L}{\sim} b$ iff $a^{-1}b \in G_0$, then observe

1) $a \overset{L}{\sim} a$

2) $b \overset{L}{\sim} a$ $(b^{-1}a = (a^{-1}b)^{-1} \in G_0)$

3) $b \overset{L}{\sim} c \Rightarrow a \overset{L}{\sim} c$ $(a^{-1}c = a^{-1}bb^{-1}c \in G_0)$

$\Rightarrow \overset{L}{\sim}$ is a equivalence relation.

Denote by ${}_{G_0}[a]$ the equivalence class of $\overset{L}{\sim}$ containing a

Indeed ${}_{G_0}[a] = aG_0 =:$ Left Coset

$(b\in G_0 \Rightarrow a^{-1}ab \in G_0 \Rightarrow a \overset{L}{\sim} ab \Rightarrow ab \in {}_{G_0}[a])$

Similarly, $a \overset{r}{\sim} b$ iff $ba^{-1} \in G_0$

$\rightsquigarrow \overset{r}{\sim}$ is an equivalence relation with equivalence class $[a]_{G_0} = G_0 a =:$ Right Coset

⚠ aG_0 & $G_0 a$ are usually not subgroups of G.

Prop.

1) ${}_{G_0}[a] = [a]_{G_0} \Leftrightarrow G_0 \triangleleft G$ (normal subgroup)

2) $G_0 \triangleleft G \Rightarrow [a]_{G_0}[b]_{G_0} = [ab]_{G_0}$

This makes $\{[a]_{G_0} \mid a\in G\}$ a group.

This group is denoted by G/G_0 and called quotion group (or factor group)

Example: $G = (\mathbb{R},+)$, $G_0 = (\mathbb{Z},+)$, then $G/G_0 = ([0,1), + \bmod 1)$

$\simeq S := (\{z\in\mathbb{C} \mid |z|=1\}, \cdot) \simeq \mathbb{T}$

Prop. If $|G| =: g < \infty$, $G_0 \triangleleft G$ with $|G_0| =: g_0 \leq g$ then $|G/G_0| = g/g_0$

Def. Let G, G' be 2 groups

A homomorphism is a map $\phi : G \to G'$ s.t. (such that) $\phi(ab) = \phi(a)\phi(b)$

If ϕ is bijective, ϕ is called an isomorphism, and we write $G \simeq G'$

If $G = G'$, ϕ is called an endomorphism

and an automorphism if bijective ($G = G'$: elements & multiplication are same)

Prop. Let ϕ be a group homomorphism from G to G'

1) If G_0 a subgroup of G, then $\phi(G_0)$ is a subgroup of G'

2) $\phi(e_G) = e_{G'}$ and $\phi(a^{-1_G}) = (\phi(a))^{-1_{G'}}$

3) $Ker(\phi) = \{a \in G \mid \phi(a) = e_{G'}\}$ is normal subgroup of G,

and $G/Ker(\phi)$ is isomorphic to $\phi(G)$

the isomorphism $\tilde{\phi}([a]_{Ker(\phi)}) := \phi(a)$