Spring Semester AY 2018

Special Mathematics Lecture

Groups and their representations

More on Roots

of Complex Semi simple Lie Algebra

(mainly from the book of Hall, chap 8)

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Definition: A root system is a finite-dimensional real vector space *E* with an inner product $\langle\cdot,\cdot\rangle$, together with a finite collection *R* of nonzero vectors in *E* satisfying the following properties:

1. span $(\alpha|\alpha \in R) = E$.
2. If $\alpha \in R$, then $-\alpha \in R$.
3. If $\alpha \in R$, then the only multiples of α in *R* are α and $-\alpha$.
4. If $\alpha, \beta \in R$, then so is $\omega_\alpha \cdot \beta$, where ω_α is called the linear transformation of *E* and is defined by:

$$\omega_\alpha \cdot \beta \coloneqq \beta - 2\frac{\langle\beta, \alpha\rangle}{\langle\alpha, \alpha\rangle}\alpha, \qquad \beta \in E$$

5. $2\frac{\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z},\ \forall\ \alpha, \beta \in R$.

The dimension of *E* is called the rank of the root system and *R* is the set of all roots.

Note: The root systems satisfying these five properties are called reduced root systems. The systems satisfying only properties 1, 2, 4 and 5 but not 3 are called unreduced root systems.

One observes that ω_α is the reflection about the hyperplane perpendicular to α:

1. $\omega_\alpha \cdot \alpha = -\alpha$.
2. $\omega_\alpha \cdot \beta = \beta, \forall\, \beta \perp \alpha$.

Thus, ω_α is an orthogonal transformation of *E* with determinant -1.

Definition: If (E, R) is a root system, then the Weyl group W of *R* is the subgroup of the orthogonal group of *E* generated by the reflections $\omega_\alpha, \alpha \in R$.

Note: Since ω_α maps *R* onto itself, every element ω of W maps *R* onto itself.

Proposition (1): Let (*E*, *R*) and (*F*, *S*) be root systems. Consider the vector space $E \oplus F$ with the natural inner product determined by the inner products on *E* and *F*. Then, $R \cup S$ is a root system in $E \oplus F$. It is called the direct sum of *R* and *S*.

Definition: A root system (*E*, *R*) is called reducible if there exists an orthogonal decomposition $E = E_1 \oplus E_2$ with dim $E_1 > 0$ and dim $E_2 > 0$, such that every element of *R* is either in E_1 or in E_2. Otherwise, it is called irreducible.

If (E, R) is reducible, one observes that the set $R_1 := \{\alpha \in R | \alpha \in E_1\}$ is a root system in E_1 and the set $R_2 := \{\alpha \in R | \alpha \in E_2\}$ is a root system in E_2.

Definition: Two root systems (E, R) and (F, S) are said to be equivalent if there exists an invertible linear transformation $A: E \to F$ such that A maps R onto S and such that:

$$A(\omega_\alpha \cdot \beta) = \omega_{A\alpha} \cdot A\beta, \forall \alpha \in R \text{ and } \beta \in E$$

Such a map A is called an equivalence.

Note: The linear map A only preserves reflections about the roots, not inner products in general.

Example:

1. If A is an orthogonal transformation of E to F and it takes R onto S, then A is an equivalence.
2. If $E = F$ and $S = \lambda R$ with $\lambda \in \mathbb{R}^*$, then $A = \lambda \mathbf{1}$ is an equivalence.

Proposition (2): Let α, β be roots, and α is not a multiple of β, and $\langle \alpha, \alpha \rangle \geq \langle \beta, \beta \rangle$. Then, one of the following case happens:

1. $\langle \alpha, \beta \rangle = 0$.
2. $\langle \alpha, \alpha \rangle = \langle \beta, \beta \rangle$ and the angle between α and β is 60° or 120°.
3. $\langle \alpha, \alpha \rangle = 2\langle \beta, \beta \rangle$ and the angle between α and β is 45° or 135°.
4. $\langle \alpha, \alpha \rangle = 3\langle \beta, \beta \rangle$ and the angle between α and β is 30° or 120°.

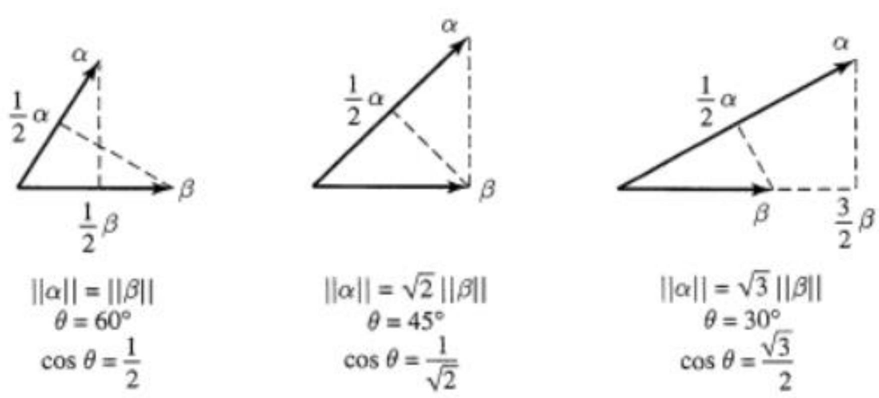

Figure 1. Allowed angles and length ratio (cases of acute angle)

Corollary (1): Let α, β be roots and $\langle\alpha, \alpha\rangle \geq \langle\beta, \beta\rangle$:

1. If the angle between them is strictly acute, then $\alpha - \beta$ and $\beta - \alpha$ are roots.
2. If the angle between them is strictly obtuse, then $\alpha + \beta$ is a root.

Proof

We will only prove for the first statement, the other can be done similarly.

From Proposition (2) and Figure 1, it can be easily shown that the projection of β onto α is $\frac{1}{2}\alpha$ in all of three cases (2,3 and 4).

Then, one has:

$$\omega_\alpha \cdot \beta = \beta - 2\frac{\langle\beta, \alpha\rangle}{\langle\alpha, \alpha\rangle}\alpha = \beta - 2\frac{\langle\frac{1}{2}\alpha, \alpha\rangle}{\langle\alpha, \alpha\rangle}\alpha = \beta - 2\cdot\frac{1}{2}\frac{\langle\alpha, \alpha\rangle}{\langle\alpha, \alpha\rangle}\alpha = \beta - \alpha$$

Since α, β are roots, by Property 4 in the definition of a root system, $\omega_\alpha \cdot \beta$, or $\beta - \alpha$ is also a root. By Property 2, so is $\alpha - \beta$. ■

Definition: A subset Δ of R is called a base for R if the following conditions hold:

1. Δ is a basis for E as a vector space.
2. Each root α in R can be expressed as a linear combination of elements of Δ with integer coefficients and such that all the nonzero coefficients have the same sign.

Roots of which the coefficients are non-negative are called positive roots (relative to the base Δ) and the set of them is denoted R^+. The others are called negative roots.

Elements of Δ are called positive simple roots.

Note:

1. It is not obvious that such a base always exists but there exists indeed a constructive method for exhibiting such a base.
2. The expansion of α in terms of elements of Δ is unique since Δ is a basis for E.

Proposition (3): Let α, β be distinct elements of a base Δ for R. Then, $\langle \alpha, \beta \rangle \leq 0$.

<u>*Proof*</u>

Assume $\langle \alpha, \beta \rangle > 0$. (*)

Since $\alpha \neq \beta$, the angle between them is strictly acute.

On the one hand, by Corollary (1), then $\gamma \coloneqq \alpha - \beta$ is also a root.

On the other hand, the unique expansion of γ in terms of elements of Δ has one positive coefficient and one negative coefficient. Therefore, γ is not a root.

$\Rightarrow$ contradiction.

So, the assumption (*) is wrong. Then, $\langle \alpha, \beta \rangle \leq 0$. ■