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1 Since $su(N)$ is a group, the identity I is an element ($\det I = 1$)
Also being a Lie group, there must be elements of the group which are closed to I. Let us consider an element U which differs from the identity an infinitesimal amount.

$$U = I + \varepsilon X \qquad \varepsilon \ll 1$$

Since U is unitary, to the first order of ε

$$U^* U = I + \varepsilon (X + X^*) + O(\varepsilon^2) = I$$

we choose an arbitrary ε, thus $X + X^* = 0$ or $X^* = -X$.

A finite group transformation can be obtained by writing the transform parameter $\alpha = n\varepsilon$ for n is large and applying n infinitesimal transformations

$$U = \left(I + \frac{\alpha}{n} X \right)^n \to e^{\alpha n} \quad \text{as } n \to \infty$$

So U can be written as $e^{\alpha X}$. Moreover,

$$\det U = \det e^{\alpha X} = e^{\alpha \operatorname{Tr} X} = 1$$

$$\Rightarrow \operatorname{Tr} X = 0 \quad \text{for an arbitrary } \alpha.$$

So X is skew-hermitian and traceless matrix.

2) An arbitrary $N \times N$ complex matrix is defined by $2N^2$ real parameters. Since X must be skew hermitian $X_{ij} = -\overline{X_{ji}}$ which reduces N^2 constraints that the elements must satisfy or N^2 real parameters are needed. Lastly, X is traceless matrix, this gives one more constraint. Thus X is defined by $N^2 - 1$ real parameters or $su(N)$ is of dimension $N^2 - 1$.

3) We know that the basis of $su(N)$ are linearly independent over $\mathbb{R}$, which means that

$$\sum_{i=1}^{N^2-1} \alpha_i X_i = 0 \iff \alpha_i = 0 \qquad \text{for } \alpha_i \in \mathbb{R}.$$

No we want to show that they are also linearly independent over $\mathbb{C}$. Indeed, Given $C_j = \alpha_j + i\beta_j$; $\alpha_J, \beta_J \in \mathbb{R}$. We want to show that

$$\sum_{J=1}^{N^2-1} (\alpha_J + i\beta_J) X_J = 0 \Leftrightarrow C_J = 0 \quad (1)$$

Taking the conjugate of the LHS, we get:

$$\sum_{J=1}^{N^2-1} (\alpha_J - i\beta_J) X_J^* = 0$$

From the previous part, we know that $X_J^* = -X_J$, it follows that

$$\sum_{J=1}^{N^2-1} (\alpha_J - i\beta_J) X_J = 0 \quad (2).$$

Taking $(1)+(2) \Leftrightarrow \sum_{J=1}^{N^2-1} 2\alpha_J X_J = 0$

since X_Js are linearly independent over $\mathbb{R} \Rightarrow \alpha_J = 0 \quad \forall J = 1, .., N^2-1$.

Also taking $(1)-(2) \Leftrightarrow \sum_{J=1}^{N^2-1} 2 i \beta_J X_J = 0.$

$$\Leftrightarrow \sum_{J=1}^{N^2-1} \beta_J X_J = 0$$

Similarly, this implies that $\beta_J = 0 \quad \forall J = 1, .. N^2-1$.

Thus we conclude that basis of $su(N)$ are also linearly independent over $\mathbb{C}$.

2. a) Show $K(Y_i, Y_J) = \sum_{k,r} C_{ir}^k C_{Jk}^r$

we have

$$(ad_{Y_i} ad_{Y_J}) Y_r = ad_{Y_i}([Y_J, Y_r])$$
$$= ad_{Y_i}\left(\sum_k C_{Jr}^k Y_k\right)$$
$$= \sum_k C_{Jr}^k ad_{Y_i}(Y_k) \quad \text{adjoint rep. properties}$$
$$= \sum_{k,n} C_{Jr}^k C_{ik}^n Y_n$$

Because the killing form takes the trace of $ad_{Y_i} ad_{Y_J}$, we only interested in the coefficients with which Y_r appears in the sum hence set $n = r$, we obtain

$$K(Y_i, Y_J) = \sum_{k,r} C_{Jr}^k C_{ik}^n$$

b) Show $K([X, Y], Z) = K(X, [Y, Z]) \quad \forall X, Y, Z \in \mathcal{L}$.

we have $L.h.s = Tr(ad_{[X,Y]} ad_Z) = Tr(ad_X ad_Y ad_Z - ad_Y ad_X ad_Z)$

but $Tr(AB) = Tr(BA)$, so

$$= Tr(ad_X ad_Y ad_Z - ad_X ad_Z ad_Y) = Tr(ad_X, ad_{[Y,Z]})$$