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Tensor Product

Let $\mathcal{H}$ be a Hilbert space with inner product $\langle \cdot , \cdot \rangle_{\mathcal{H}}$ and
$\mathcal{K}$ be a Hilbert space with inner product $\langle \cdot , \cdot \rangle_{\mathcal{K}}$.

Def. If V, W, U are complex vector space, a map

$\varphi : V \times W \longrightarrow U$ is bilinear if

$$\varphi(x_1 + x_2, y_1) = \varphi(x_1, y_1) + \varphi(x_2, y_1)$$
$$\varphi(x_1, y_1 + y_2) = \varphi(x_1, y_1) + \varphi(x_1, y_2)$$
$$\varphi(\lambda x_1, y_1) = \varphi(x_1, \lambda y_1) = \lambda \varphi(x_1, y_1)$$

for all $x_i \in V$ $y_i \in W$ and $\lambda \in \mathbb{C}$

Def. A tensor product of $\mathcal{H}$ with $\mathcal{K}$ is a Hilbert space $\mathcal{P}$, together with a bilinear map $\varphi : \mathcal{H} \times \mathcal{K} \to \mathcal{P}$ s.t.

(1) the set of all vectors $\varphi(x, y)$ $(x \in \mathcal{H}, y \in \mathcal{K})$ forms a total subset of $\mathcal{P}$, that is, its closed linear span is equal to $\mathcal{P}$

(2) $\langle \varphi(x_1, y_1), \varphi(x_2, y_2) \rangle_{\mathcal{P}} = \langle x_1, x_2 \rangle \langle y_1, y_2 \rangle$

for $x_1, x_2 \in \mathcal{H}$, $y_1, y_2 \in \mathcal{K}$

We refer to the pair $(\mathcal{F}, \varphi)$ as the tensor product

If $(\mathcal{F}, \varphi)$ is a tensor product of $\mathcal{H}$ with K,

it writes $x \otimes y$ in place of $\varphi(x, y)$. and $\mathcal{H} \otimes K$

in place of $\mathcal{F}$.

By definition

- $\mathcal{H} \otimes K$ is a Hilbert space
- a map $(x, y) \mapsto x \otimes y$ of $\mathcal{H} \times K$ into $\mathcal{H} \otimes K$

s.t.
$$(x_1 + x_2) \otimes y_1 = x_1 \otimes y_1 + x_2 \otimes y_1$$
$$x_1 \otimes (y_1 + y_2) = x_1 \otimes y_1 + x_1 \otimes y_2$$
$$(\lambda x_1) \otimes y_1 = x_1 \otimes (\lambda y_1) = \lambda (x \otimes y)$$

- the vectors $x \otimes y$ form a total subset of $\mathcal{H} \otimes K$
- $\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle_{\mathcal{H} \otimes K} = \langle x_1 ; x_2 \rangle_{\mathcal{H}} \langle y_1, y_2 \rangle_{K}$

We shall denote by $\mathcal{H} \odot K$ the linear subspace of $\mathcal{H} \otimes K$ generated by the vectors $x \otimes y$.

Condition (1) is equivalent to the assertion that $\mathcal{H} \odot K$ is a dense linear subspace of $\mathcal{H} \otimes K$.

Thm. (Universal property)

There exists $\mathcal{H} \odot \mathcal{K}$ with a bilinear map $\otimes : \mathcal{H} \times \mathcal{K} \longrightarrow \mathcal{H} \odot \mathcal{K}$ such that for any vector space $\mathcal{E}$ and for any bilinear map $\psi : \mathcal{H} \times \mathcal{K} \longrightarrow \mathcal{E}$ there uniquely exists a linear map $f : \mathcal{H} \odot \mathcal{K} \longrightarrow \mathcal{E}$ such that $\psi = f \circ \otimes$