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Universal enveloping algebras and Casimir operators.

In order to construct Casimir operators on a Lie algebra, we introduce universal enveloping algebras.

Let F be a field and let $\mathfrak{g}$ be a Lie algebra. (allowed to be infinite dimensional.)

The universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ is a associative algebra with 1 which contains $\mathfrak{g}$ as a subspace and the commutator of $U(\mathfrak{g})$ corresponds with the Lie bracket on $\mathfrak{g}$.

Def (universal enveloping algebra of $\mathfrak{g}$)

Let $T(\mathfrak{g}) := \bigoplus_{i=0}^{\infty} \left(\bigotimes_i \mathfrak{g} \right)$ be a tensor algebra of $\mathfrak{g}$

$\left(\bigotimes_i \mathfrak{g} := \underbrace{\mathfrak{g} \otimes \cdots \otimes \mathfrak{g}}_{i \text{ copies}} \quad \bigotimes_1 \mathfrak{g} = \mathfrak{g}, \ \bigotimes_0 \mathfrak{g} = F \right)$

and let I be a two-sided ideal generated by

$x \otimes y - y \otimes x - [x, y] \quad (x, y \in \mathfrak{g})$.

We set the universal enveloping algebra of $\mathfrak{g}$.

$$U(\mathfrak{g}) := T(\mathfrak{g}) / I$$

and let $\pi : T(\mathfrak{g}) \longrightarrow U(\mathfrak{g})$ be the canonical homomorphism.

<u>Remark</u> By Poincaré-Birkhoff-Witt theorem and its corollary (see [1]), the canonical homomorphism $i := \pi|_{\mathfrak{g}} : \mathfrak{g} \longrightarrow U(\mathfrak{g})$ is injective.

Then, we can embed $\mathfrak{g}$ in $U(\mathfrak{g})$ as a subspace.

And by another corollary, we have a basis for $U(\mathfrak{g})$.
(see [1])

If $(x_1, x_2, \cdots)$ is a ordered basis for $\mathfrak{g}$, then

$\{ x_{l(1)} \cdot \cdots \cdot x_{l(m)} \mid m \in \mathbb{N},\ l(1) \leq l(2) \leq \cdots \leq l(m) \}$ is a basis for $U(\mathfrak{g})$.

(This basis is called PBW basis.)

Then, even when $\mathfrak{g}$ is finite dimensional, $U(\mathfrak{g})$ is not in general finite dimensional.

The universal enveloping algebra $U(\mathfrak{g})$ and canonical homomorphism $i : \mathfrak{g} \to U(\mathfrak{g})$ has the following universal property: (see [1])

$^\forall A$: an associative F-algebra with 1, $^\forall \varphi : \mathfrak{g} \to A$: F-linear map

$\varphi([x, y]) = \varphi(x)\varphi(y) - \varphi(y)\varphi(x)$

$\exists! \tilde{\varphi} : U(\mathfrak{g}) \to A$: algebra homomorphism sending 1 to 1

s.t. $\varphi = \tilde{\varphi} \circ i$

$$
\begin{tikzcd}
\mathfrak{g} \arrow[r, "\varphi"] \arrow[d, "i"'] \arrow[dr, phantom, "\circlearrowleft"] & A \\
U(\mathfrak{g}) \arrow[ur, dashed, "\tilde{\varphi}"'] &
\end{tikzcd}
$$

e.g. $U(\mathfrak{sl}(2,\mathbb{C})) \cong \mathbb{C}\langle x,y,z\rangle / (zx-xz-2x,\ zy-yz+2y,\ xy-yx-z)$

$$X=\begin{pmatrix}0&1\\0&0\end{pmatrix} \quad Y=\begin{pmatrix}0&0\\1&0\end{pmatrix} \quad H=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$$\updownarrow \qquad\qquad \updownarrow \qquad\qquad \updownarrow$$

$$x \qquad\qquad y \qquad\qquad z$$

where $\mathbb{C}\langle x,y,z\rangle$ is a non-commutative polynomial ring (free algebra)

Now, let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra.

and let $\{T_i\}$ be a basis for $\mathfrak{g}$ with $[T_j, T_k] = \sum_i f^i_{jk} T_i$

The Casimir operators are elements of the center $Z(U(\mathfrak{g}))$ of $U(\mathfrak{g})$. (see [2])

$$C_n := \sum_{i_1,\dots,i_n} d^{i_1\cdots i_n} T_{i_1}\cdots T_{i_n}$$

with $d^{i_1\cdots i_k}$ suitable symmetric invariant tensors of the adjoint representation

where symmetric tensor means $\forall \sigma \in \mathfrak{S}_n \ d^{\sigma(i_1)\cdots\sigma(i_n)} = d^{i_1\cdots i_n}$

and invariant under adjoint representation means.

$$0 = f^{i_1}_{k\ell}\, d^{\ell i_2\cdots i_n} + \cdots + f^{i_n}_{k\ell}\, d^{i_1\cdots i_{n-1}\ell}$$

In particular, quadratic Casimir operator is the element.

$$C_2 := \sum_{i,j} k^{ij} T_i T_j$$

with k^{ij} is the inverse matrix of Killing form. (Killing form is symmetric and invariant under adjoint rep.)

e.g. the quadratic Casimir operator of $sl(2,\mathbb{C})$

⚠ Conventions and normalization are different from the lecture.

it is known that the Killing form of $sl(2,\mathbb{C})$ is

$$K(X,Y) = 4\,\mathrm{tr}(XY) \quad (X, Y \in sl(2,\mathbb{C}))$$

Now, $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

are a basis for $sl(2,\mathbb{C})$ $[X,Y] = H$, $[H,X] = 2X$, $[H,Y] = -2Y$

Then, $K(X,Y) = 4$, $K(X,H) = K(Y,H) = 0$,

$K(H,H) = 8$ $K(X,X) = K(Y,Y) = 0$

Thus, $K = (X\ Y\ H) \begin{pmatrix} 0 & 4 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 8 \end{pmatrix} \begin{pmatrix} X \\ Y \\ H \end{pmatrix}$

Then $C_2 = \frac{1}{8}H^2 + \frac{1}{4}XY + \frac{1}{4}YX = \frac{1}{8}H^2 + \frac{1}{4}H + \frac{1}{2}YX$

C_2 is a element of the center $U(sl(2,\mathbb{C}))$. Indeed,

$$\begin{aligned}
[C_2, H] &= \frac{1}{8}[H^2,H] + \frac{1}{4}[H,H] + \frac{1}{2}[YX,H] = \frac{1}{2}(YXH - HYX) \\
&= \frac{1}{2}(YXH - YHX + YHX - HYX) \\
&= \frac{1}{2}(Y[X,H] + [Y,H]X) = -YX + YX = 0
\end{aligned}$$

$$\begin{aligned}
[C_2, X] &= \frac{1}{8}[H^2,X] + \frac{1}{4}[H,X] + \frac{1}{2}[YX,X] \\
&= \frac{1}{8}(HHX - XHH) + \frac{1}{2}X + \frac{1}{2}(YXX - XYX)
\end{aligned}$$

$$= \frac{1}{8}(HHX - HXH + HXH - XHH) + \frac{1}{2}X + \frac{1}{2}(YXX - XYX)$$

$$= \frac{1}{8}(H[H,X] + [H,X]H) + \frac{1}{2}X + \frac{1}{2}([Y,X]X)$$

$$= \frac{1}{4}HX + \frac{1}{4}XH + \frac{1}{2}X - \frac{1}{2}HX$$

$$= \frac{1}{4}[X,H] + \frac{1}{2}X = -\frac{1}{2}X + \frac{1}{2}X = 0$$

Similarly, $[C_2, Y] = 0$. Then $C_2 \in Z(U(\mathfrak{g}))$

Thm. (see [3])

Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra of rank r.
The center $Z(U(\mathfrak{g}))$ is isomorphic to polynomial algebra $F[C^{(i)}]$ $(i = 1, \cdots, r)$. Therefore, the number of algebraically independent Casimir operators is equal to the rank.

Reference

[1] James E. Humphreys, Introduction to Lie Algebras and Representation Theory, (Springer 1972)

[2] Jürgen Fuchs, Christoph Schweigert, Symmetries, Lie Algebras and Representations, (Cambridge University Press, 2003)

[3] Xavier Bekaert, Universal enveloping algebras and some application in physics. http://www.ulb.ac.be/sciences/ptm/pmif/Rencontres/Modavel/Xavier.pdf