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Review of the solution of (ii), exercise 3 by Y.J. Ng.

I don't think your computation of φ_x and φ_y is correct. Let me give you a complete solution of (ii) using the implicit function theorem approach.

Solution of (ii), Exercise 3:

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a smooth map (or sufficiently differentiable).

Denote by $\mathcal{L}_r$ the r-level set of f, i.e.,

$$\mathcal{L}_r = \{ x \in \mathbb{R}^n \mid f(x) = r \}.$$

In our case, as $f : \mathbb{R}^3 \to \mathbb{R}$ defined as in the exercise satisfying the conditions at $(1,1,1)$, (that is, $\partial_z f(1,1,1) \neq 0$ and $f(1,1,1) = 0$), the implicit function theorem states that (think about it):

there exists a neighbourhood U of $(1,1)$ in $\mathbb{R}^2$ and $\phi : U \longrightarrow \mathbb{R}$, such that locally (i.e., on a neighbourhood V of $(1,1,1)$ in $\mathbb{R}^3$)

$$\mathcal{L}_0 = \{ (x, y, \phi(x,y)) \} \quad \text{for } (x,y) \in U. \quad (*)$$

To find the tangent plane to the surface defined by $f(x,y,z) = 0$ at $(1,1,1)$, it suffices to find the expression of the tangent plane of the graph of $\phi(x,y)$ at $(1,1)$, according to the result $(*)$.

Now we are going to find $\underbrace{\phi_x}_{= \partial_x \phi}$ and $\underbrace{\phi_y}_{= \partial_y \phi}$ at $(1,1)$. We use the chain rule for this purpose.

We define $\hat{f}(x,y) := f(x, y, \phi(x,y))$ to be a function $U \subset \mathbb{R}^2 \to \mathbb{R}$. However, $(*)$ suggests that $\hat{f} \equiv 0$ on U, which implies that

$$\partial_x \hat{f} = \partial_y \hat{f} = 0.$$

On the other hand, $\partial_x \hat{f}$ and $\partial_y \hat{f}$ can be obtained using the chain rule.

In fact, $\hat{f} : \mathbb{R}^2 \to \mathbb{R}$ can be regarded as the composition of two functions:

$$\mathbb{R}^2 \xrightarrow{g} \mathbb{R}^3 \xrightarrow{f} \mathbb{R},$$

where f is given, and g is in fact:

$$\begin{pmatrix} x \\ y \end{pmatrix} \xmapsto{g} \begin{pmatrix} x \\ y \\ \phi(x,y) \end{pmatrix}$$

Applying the chain rule, we have (the product of the Jacobian matrices)

$$(0,0) = J_{F}(x,y) = J_f(g(x,y)) \cdot J_g(x,y) =$$

another notation for $\quad = \partial_x f \quad = \partial_y f \quad = \partial_z f$

$$\big(f_x(x,y,\phi(x,y)),\ f_y(x,y,\phi(x,y)),\ f_z(x,y,\phi(x,y))\big) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \phi_x(x,y) & \phi_y(x,y) \end{pmatrix}$$

$$= \big(f_x(x,y,\phi(x,y)) + f_z(x,y,\phi(x,y))\,\phi_x(x,y),\ f_y(x,y,\phi(x,y)) + f_z(x,y,\phi(x,y))\,\phi_y(x,y)\big) \quad (**)$$

Now evaluate $(**)$ at $(x,y) = (1,1)$. Note that $f_z(1,1,1) \neq 0$, we have:

$$\begin{cases} \phi_x(1,1) = -f_x(1,1,1)/f_z(1,1,1) \\ \phi_y(1,1) = -f_y(1,1,1)/f_z(1,1,1) \end{cases} \quad (***)$$

Now we only need to compute f_x and f_y. We have:

$$\begin{cases} f_x = 2x - y^3 \\ f_y = -3xy^2 - 2yz \\ f_z = -y^2 + 3z^2 \end{cases} \Longrightarrow \begin{cases} f_x(1,1,1) = 1 \\ f_y(1,1,1) = -5 \\ f_z(1,1,1) = 2 \end{cases},$$

then $(***)$ gives us that $\phi_x(1,1) = -1/2$ and $\phi_y(1,1) = 5/2$.

The expression of the tangent plane of $\phi(x,y)$ at $(1,1)$ is given by:

$$z = \phi_x(1,1)(x-1) + \phi_y(1,1)(y-1) + 1$$

$$\Longleftrightarrow \quad x - 5y + 2z + 2 = 0.$$

$\square$

Since the tangent plane at $(1,1)$ is the linear term in the Taylor expansion, one has

$$z = \phi(x,y) \cong \phi(1,1) + \begin{pmatrix} x-1 \\ y-1 \end{pmatrix} \cdot \nabla\phi(1,1)$$
$$= 1 + (x-1)[\partial_x \phi](1,1) + (y-1)[\partial_y \phi](1,1).$$