$aG_0 = G_0 a \Leftrightarrow G_0 \lhd G$:

Proof.

1) $G_0 \lhd G \Rightarrow aG_0 = G_0 a$.:

$\forall a \in G;\ \forall x \in aG_0;\ \forall y \in G_0 a;$

$a^{-1}x \in G_0;\ ya^{-1} \in G_0.$

$x' := a^{-1}x;\ y' := ya^{-1};$

$s := ax'a^{-1} \in G_0;\ t := a^{-1}y'a \in G_0.$

$ax' = ax'a^{-1}a = sa = x$

$y'a = aa^{-1}y'a = at = y$

$\because s \in G_0,\ t \in G_0.$

$\therefore sa \in G_0 a;\ at \in aG_0.$

$\because \forall x \in aG_0,\ \exists sa \in G_0 a$ ST $sa = x$

...

$\therefore G_0 \lhd G \Rightarrow aG_0 = G_0 a.$

del.

2) $aG_0 = G_0 a \Rightarrow G_0 \lhd G.$

$aG_0 = G_0 a \Rightarrow$

$\forall s \in G_0,\ \exists s' \in G_0$ ST.

$sa = as'$

$\Downarrow$

$a^{-1}sa = a^{-1}as' = s'$

$\therefore \forall s \in G_0,\ a^{-1}sa \in G_0 \therefore G_0 \lhd G$

for 1) and 2); $aG_0 = G_0 a \Leftrightarrow G_0 \lhd G$

1). $\forall a \in G;\ \forall L \in aG_0;\ \exists l \in G_0$ ST $al = L.$

by def, $\exists r = ala^{-1} \in G_0.$

$R := r \cdot a = ala^{-1}a = al = L.;\ R \in G_0 a$

$\therefore \forall L \in aG_0,\ \exists R \in G_0 a$ ST $R = L.$

$\therefore L \subset R.$

~ Same Inversely ~

$R \subset L.$

$\therefore G_0 \lhd G \Rightarrow aG_0 = G_0 a.$

$[a]_{G_0}[b]_{G_0} = G_0 a G_0 b = G_0 G_0 ab$

$\because \forall a,b \in G_0;\ ab \in G_0$

$\therefore G_0 G_0 = G_0.$

$\therefore G_0 G_0 ab = G_0 ab = [ab]_{G_0}$

$\phi: G \to G'$, ϕ is homomorphism.

If G_0 is subgroup of $G \Rightarrow \phi G_0$ is subgroup of G'

⓪ $\forall a', b' \in \phi G_0;\ \exists a, b \in G_0$ ST $\phi(a) = a';\ \phi(b) = b'.$

Thus, for $ab \in G_0;\ \phi(ab) = \phi(a)\phi(b) = a'b' \in \phi G_0.$

① $\phi(e_G) \in \phi G_0$, and $\forall k' \in \phi G_0,\ \exists k \in G_0$ ST $\phi k = k'.$

$\phi(e_G) \cdot k' = \phi(e)\phi(k) = \phi(ek) = \phi(k) = k'.$

$\therefore \phi(e_G) = e_{G'}$ #2).1

② $\forall a, a^{-1_G} \in G_0$:

$a \cdot a^{-1_G} = e_G$

$\phi(a \cdot a^{-1_G}) = \phi(a) \cdot \phi(a^{-1_G})$

$\phi(e_G) = e_{G'}$

$\phi(a) \cdot \phi(a^{-1_G}) = e_{G'}$

$\phi(a)^{-1_{G'}}\phi(a) \cdot \phi(a^{-1_G}) = \phi(a)^{-1_{G'}} \cdot e_{G'}$

$\phi(a^{-1_G}) = \phi(a)^{-1_{G'}}$ #2).2

For ⓪, ①, ②; $\phi(G_0) \in G'$, G_0 is subgroup of $G \Rightarrow \phi G_0$ is subgroup of G'

$\ker\phi = \{a \in G \mid \phi a = e_{G'}\}$ is a normal subgroup of G:

⓪ $\forall a,b \in \ker\phi;\ \phi(a) = e_{G'},\ \phi(b) = e_{G'} \Rightarrow \phi(ab) = e_{G'} \cdot e_{G'} = e_{G'}$

$\Downarrow$

$ab \in \ker\phi.$

① $\phi(e_G) = e_{G'}$ #2).1

② $\forall a \in \ker\phi,\ \exists a^{-1} \in G$ ST $aa^{-1} = e_G.$

$\phi(aa^{-1}) = \phi(a)\phi(a^{-1}) = e_{G'}\phi(a^{-1}) = \phi(e_G) = e_{G'}$

$\phi(a^{-1}) = e_{G'}.$

$\therefore a^{-1} \in \ker\phi.$

$\therefore \ker\phi$ is a subgroup of $G.$ < $\because \ker\phi \subset G.$

$\forall c \in G;\ \forall n \in \ker\phi;$

$\phi(cnc^{-1_G}) = \phi(c)e_{G'}\phi(c^{-1}) \overset{\#2).2}{=} \phi(c)\phi(c)^{-1_{G'}} = e_{G'}$

$\therefore cnc^{-1} \in \ker\phi$

$\therefore \ker\phi \lhd G$

$\tilde{\phi}([a]\ker\phi) := \phi(a)$ Proof for homomorphism: $\forall a, b \in G$; $\tilde{\phi}([a]G[b]G) = \phi([ab]G) = \phi(ab) = \phi(a)\phi(b)$
$= \phi([a]G)\,\phi([b]G)$

$\ker\tilde{\phi} = \{[a]\ker\phi \mid \tilde{\phi}([a]\ker\phi) = e\} = \{[a]\ker\phi \mid a \in \ker\phi\}$ $\because \forall a, b \in \ker\phi,\ ab \in \ker\phi$; $[a]\ker\phi = \ker\phi = e\ker\phi$.

$\forall a, b \in G$

$\therefore \forall a \in G; a \neq e;\ \tilde{\phi}([a]\ker\phi) \neq e'$

If $\|G/\ker\phi\| > 1$ $\forall a, b \in G/\ker\phi$ st $a \neq b$; $c := ab^{-1} \in G/\ker\phi$.

$\tilde{\phi}([a]\ker\phi) = \tilde{\phi}([cb]\ker\phi) = \phi(a) = \phi(c)\phi(b)$

$\because c \neq e;\ \therefore \phi(c) \neq e'$

$\therefore \phi(a) \neq \phi(b)$; $\phi([a]\ker\phi) \neq \phi([b]\ker\phi)$. $\therefore \tilde{\phi}$ is injective if $\|G/\ker\phi\| > 1$.

Else if $\|G/\ker\phi\| = 1$ Injective anyway.

$\therefore \tilde{\phi}$ is injective.

$\forall\ m' \in \phi(G)$; $\exists\, m \in G$ ST $\phi(m) = m'$.

$\therefore\ \exists\, q \in G/\ker\phi$ ST $m \in q$

$\because \forall a \in G; \forall a_i \in a\ker\phi$; $\exists k_i \in \ker\phi$ ST $a \cdot k_i = a_i$

$\phi(a') = \phi(a \cdot k_i) = \phi(a)\,\phi(k_i) = \phi(a) \cdot e' = \phi(a)$

$\therefore \forall a_i \in a\ker\phi$, $\tilde{\phi}([a]\ker\phi) = \phi(a) = \phi(a_i)$

$\because m \in q$; $q \in G/\ker\phi$;

$\therefore \tilde{\phi}(q) = \phi(m) = m'$

$\therefore \tilde{\phi}$ is surjective, thus bijective.

$\because \tilde{\phi}$ is homomorphism;

$\therefore \tilde{\phi}$ is isomorphism. $\square$