ZHANG NUOZHOU

1) If $\{Y_1, \ldots, Y_n\}$ is a basis of $\mathcal{L}$ with $[Y_i, Y_j] = \sum_{k=1}^{n} C_{ij}^{k} Y_k$

Then $g_{ij} = K(Y_i, Y_j) = \sum_{k,r} C_{ir}^{k} C_{jk}^{r}$

2) $K([X,Y],Z) = K(X,[Y,Z]) \quad \forall X, Y, Z \in \mathcal{L}$

3) $[K(X,Y)=0 \ \forall Y \in \mathcal{L} \Rightarrow X=0] \Leftrightarrow \det(g_{ij}) \neq 0$

Proof: 1)

$$g_{ij} = K(Y_i, Y_j) = Tr(ad_{Y_i} ad_{Y_j})$$
$$= \sum_m \langle Y_m, ad_{Y_i} ad_{Y_j}(Y_m) \rangle$$
$$= \sum_m \langle Y_m, ad_{Y_i} \sum_n C_{jm}^{n} Y_n \rangle$$
$$= \sum_{m,n} C_{jm}^{n} \langle Y_m, ad_{Y_i}(Y_n) \rangle$$
$$= \sum_{m,n} C_{jm}^{n} \langle Y_m, \sum_l C_{in}^{l} Y_l \rangle$$
$$= \sum_{m,n,l} C_{jm}^{n} C_{in}^{l} \delta_{ml}$$
$$= \sum_{m,n} C_{jm}^{n} C_{in}^{m}$$

2) $$K([X,Y],Z) = Tr(ad_{[X,Y]} ad_Z)$$
$$= Tr(ad_X ad_Y ad_Z - ad_Y ad_X ad_Z)$$

due to the cyclicity of the trace.

$$= Tr(ad_X ad_Y ad_Z - ad_X ad_Z ad_Y)$$
$$= Tr(ad_X (ad_Y ad_Z - ad_Z ad_Y))$$
$$= Tr(ad_X ad_{[Y,Z]})$$
$$= K(X,[Y,Z])$$

3) Since $\mathcal{L}$ is a vector space with $\{Y_1, \ldots, Y_n\}$ as basis, construct the dual space $\mathcal{L}^*$ with basis $\{Y_1^*, \ldots, Y_n^*\}$ such that $Y_i^*(Y_j) = \delta_{ij}$ therefore $K: \mathcal{L} \to \mathcal{L}^*$ with $\varphi_X \in \mathcal{L}^*$ and $\varphi_X(Y) = K(X,Y)$

the matrix form of K is defined by $k_{ij} = \varphi_{Y_i}(Y_j) = K(Y_i, Y_j) = g_{ij}$

i) if $K(X,Y) = 0 \ \forall Y \in \mathcal{L} \Rightarrow X = 0$, then $\ker(K) = 0 \Rightarrow$ ~~k_{ij} is~~ K is injective ~~$\Rightarrow k_{ij} = g_{ij}$ are~~ ; since $\dim(L) = \dim(L^*) \Rightarrow k_{ij}$ is invertible $\Rightarrow g_{ij}$ is invertible $\Rightarrow \det(g_{ij}) \neq 0$

ii) if $\det(g_{ij}) \neq 0 \Rightarrow K$ is injective in $\mathcal{L} \to \mathcal{L}^* \Rightarrow \ker(K) = 0$
$$\Rightarrow K(X,Y) = 0 \ \forall Y \in \mathcal{L} \Rightarrow X = 0.$$