SPARSH MISHRA

Let $\mathcal{L}$ be a lie algebra of dimension n.
Consider $\mathcal{L} \oplus \mathcal{L}$ with

$$(\lambda + i\mu)(X, Y) := (\lambda X - \mu Y, \mu X + \lambda Y) \quad \forall\ \lambda, \mu \in \mathbb{R},\ X, Y \in \mathcal{L} \quad (*)$$

Exercise: Check that this defines a complex vector space of dim n with $(X_1, 0), \ldots, (X_n, 0)$ with $\{X_1, X_2, \ldots, X_n\}$ a basis of $\mathcal{L}$

Solution: We shall first prove that $\mathcal{L} \oplus \mathcal{L}$ is a complex vector space

The following are properties of $\mathcal{L} \oplus \mathcal{L}$ (with complexification)

1) Commutativity: (This arises from the commutativity defined for direct sums)

$$(X, Y) + (Z, W) := (X+Z, Y+W)$$

since $\mathcal{L}$ is a lie algebra, it is a vector space.

$$\text{So, } (X+Z, Y+W) \overset{\text{commutativity of } \mathcal{L}}{=\!=\!=} (Z+X, W+Y)$$
$$= (Z, W) + (X, Y)$$

so, $\mathcal{L} \oplus \mathcal{L}$ (with complexification) is commutative.

2) Associativity: (This arises from the associativity of direct sums)

$$\big((X, Y) + (Z, W)\big) + (A, B)$$

$$= \big((X+Z), (Y+W)\big) + (A, B)$$

$$= \big((X+Z) + A, (Y+W) + B\big)$$

Using associativity of $\mathcal{L}$ (vector space)

$$= \big(X + (Z+A), Y + (W+B)\big)$$

$$= (X, Y) + \big((Z, W) + (A, B)\big)$$

so $\mathcal{L} \oplus \mathcal{L}$ (with complexification) is associative.

3) Scalar multiplication :

let $\lambda, \mu \in \mathbb{R}$, $\lambda + i\mu \in \mathbb{C}$

with $(X, Y) \in \mathcal{L} \oplus \mathcal{L}$

$(\lambda + i\mu)(X, Y) = (\lambda X - \mu Y, \mu X + \lambda Y)$ (using complexification map ⓕ on previous page)

$(\lambda X - \mu Y, \mu X + \lambda Y) \in \mathcal{L} \oplus \mathcal{L}$

Thus, $(\lambda + i\mu)(X, Y) \in \mathcal{L} \oplus \mathcal{L}$

4 Since $\mathcal{L} \oplus \mathcal{L}$ (with complexification) is commutative, Associative and has a well defined scalar multiplicati over $\mathbb{C}$, $\mathcal{L} \oplus \mathcal{L}$ (with complexification) is a vector space over $\mathbb{C}$.

We shall now try to find the basis of $\mathcal{L} \oplus \mathcal{L}$ (with comp.)

<u>without</u> complexification basis of $\mathcal{L} \oplus \mathcal{L}$ is

$$\{ (X_i, X_j) \in \mathcal{L} \oplus \mathcal{L} \mid i, j \in 1, 2, 3, \dots, n \}$$

This cannot be the basis of $\mathcal{L} \oplus \mathcal{L}$ with complexification. Consider the example of 2 vectors (X_1, X_2), $(-X_2, X_1)$

They are linearly independent over $\mathbb{R}$ but

$i(X_1, X_2) = (-X_2, X_1)$

$\Rightarrow$ They are <u>not</u> linearly independent over $\mathbb{C}$.

$$(-X_i, X_j) = i(X_j, X_i)$$

also, $(X_i, X_j) = (X_i, 0) + (0, X_j)$

$= (X_i, 0) + i(X_j, 0)$ for any i, j

Thus we can generate any (X_i, X_j) using a linear combination of $(X_i, 0)$ $i = 1, 2, \dots, n$ over $\mathbb{C}$

Thus $\{(X_1, 0), (X_2, 0), \dots, (X_n, 0)\}$ is a basis of $\mathcal{L} \oplus \mathcal{L}$ with complexification and is of dimension n.