Very short summary on matrices

A $n \times n$ matrix is a collection of n^2 numbers

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ a_{21} & a_{22} & \dots & a_{2N} \\ \vdots & & \dots & \vdots \\ a_{N1} & a_{N2} & \dots & a_{NN} \end{pmatrix} \equiv (a_{ij})_{i,j=1}^{n}$$

short notation

For example $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & \frac{2}{3} \end{pmatrix}$ are 2×2 matrices, and $\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, ... are 3×3 matrices.

The numbers a_{ij} can be real ($a_{ij} \in \mathbb{R}$) or complex ($a_{ij} \in \mathbb{C}$).

$\mathbb{R}$: the set of all real numbers

$\mathbb{C}$: the set of all complex numbers

The set of all $n \times n$ matrices is denoted by $M_{n \times n}(\mathbb{R})$ or $M_{n \times n}(\mathbb{C})$, if the entries are real or complex.

We define an addition of 2 matrices

$$(a_{ij})_{i,j=1}^{n} + (b_{ij})_{i,j=1}^{n} = (a_{ij} + b_{ij})_{i,j=1}^{n}$$

Example: $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ 10 & 12 \end{pmatrix}$.

We also define a product

$$(a_{ij})_{i,j=1}^{n} \quad (b_{ij})_{i,j=1}^{n} = (c_{ij})_{i,j=1}^{n}$$

with $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

Example: $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 1\cdot 5 + 2\cdot 7 & 1\cdot 6 + 2\cdot 8 \\ 3\cdot 5 + 4\cdot 7 & 3\cdot 6 + 4\cdot 8 \end{pmatrix}$

$= \begin{pmatrix} 5+14 & 6+16 \\ 15+28 & 18+32 \end{pmatrix} = \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}$.

⚠ addition and product of matrices of the same size are considered

↑ here the size is n.

Inverse For some matrix $(a_{ij})_{i,j=1}^{n}$ there exist an inverse, which means another matrix $(b_{ij})_{i,j=1}^{n}$ such that

$$(a_{ij})_{i,j=1}^{n} \quad (b_{ij})_{ij=1}^{n} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & 0 & \cdots & & 1 \end{pmatrix}$$

only 1 on the diagonal

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More simply, if A denotes a $n \times n$ matrix and if A possesses an inverse, we write A^{-1} for its inverse. It satisfies

$$A A^{-1} = A^{-1} A = \mathbb{1}$$

product of matrices

matrix $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

only 1 on the diagonal, 0 elsewhere.

The set of all $n \times n$ matrices having an inverse is denoted by $\underline{GL(n, \mathbb{R})}$ or $\underline{GL(n, \mathbb{C})}$ depending if we consider the matrices with real or complex entries.

The <u>determinant</u> is a function on $M_{n \times n}(\mathbb{C})$ taking values in $\mathbb{C}$, or from $M_{n \times n}(\mathbb{R})$ and taking values in $\mathbb{R}$.

Example : $\det \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = 1 \cdot 4 - 2 \cdot 3 = -2$

For general formula for $n = 2$ is

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11} a_{22} - a_{21} a_{12}$$

and for $n = 3$ it is

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} =$$

$$= a_{11} (a_{22} a_{33} - a_{32} a_{23}) - a_{12} (a_{21} a_{33} - a_{31} a_{23})$$

$$+ a_{13} (a_{21} a_{32} - a_{31} a_{22}) .$$

Similar formulas exist for any n but they are more and more complicated.

Note that if $A = \begin{pmatrix} a_{11} & & 0 \\ & a_{22} & \\ 0 & & \ddots \\ & & & a_{nn} \end{pmatrix}$ diagonal matrix

then $\det(A) = a_{11} a_{22} \cdots a_{nn}$.

Main properties of det :

1) $\det(AB) = \det(A) \det(B)$

2) $\det(A) \neq 0$ if and only if A^{-1} exists

⚠ $\det(A+B) \neq \det(A) + \det(B)$

↑ in general

We can now define :

$GL(n, \mathbb{C})$ all $n \times n$ matrices with complex entries and which have an inverse.	$GL(n, \mathbb{R})$ all $n \times n$ matrices with real entries and which have an inverse.
$SL(n, \mathbb{C})$ all $n \times n$ matrices with complex entries and with determinant equal to 1	$SL(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R})$ such that $\det(A) = 1\}$

Adjoint : for any matrix $(a_{ij})_{i,j=1}^{n}$ we define the adjoint matrix $(b_{ij})_{i,j=1}^{n}$ by

$$b_{ij} = \overline{a_{ji}}$$ ↖ complex conjugate

Example : if $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ then

$$A^* = \begin{pmatrix} \overline{a_{11}} & \overline{a_{21}} \\ \overline{a_{12}} & a_{22} \end{pmatrix}$$

↑ adjoint matrix

the set of $n \times n$ unitary matrices

We can now define such that

$$U(n) = \{ A \in GL(n, \mathbb{C}) \text{ s.t. } A^* = A^{-1} \}$$

the adjoint is equal to the inverse

$$SU(n) = \{ A \in U(n) \text{ s.t. } \det(A) = 1 \}.$$

Also $O(n) = \{ A \in GL(n, \mathbb{R}) \text{ s.t. } A^* = A^{-1} \}$

and $SO(n) = \{ A \in O(n) \text{ s.t. } \det(A) = 1 \}$.

Remark : for $A \in GL(n, \mathbb{R})$, since all entries are real, the adjoint of $(a_{ij})_{i,j=1}^n$ is the matrix $(b_{ij})_{i,j=1}^n$ with

$$b_{ij} = a_{ji}.$$

In this case, the adjoint is also called the transpose and is denoted by A^t.