

IV Scattering theory: stationary expressions.

IV.1 Preliminaries

We always consider $H=H^*$, $H_0=H_0^*$ in \mathcal{H} .

The wave operators have been defined with strong limits but we could use other topologies. For example,

$$W_{\pm}^{\text{weak}}(H, H_0) = \text{w-lim}_{t \rightarrow \pm\infty} E_{ac}(H) e^{itH} e^{-itH_0} E_{ac}(H_0)$$

such that $W_{\pm}^{\text{weak}}(H, H_0)^* = W_{\pm}^{\text{weak}}(H_0, H)$.

but then these operators are not always isometries.

Consider $\omega: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ s.t.

$$\int_0^{\infty} \omega(t) dt = 1$$

and

$$\mathcal{F} \left(\mathbb{R} \ni x \mapsto e^{-x} \omega(e^{-x}) \right)$$

Fourier transform

$\left\{ \psi(\cdot + \alpha) \mid \alpha \in \mathbb{R} \right\}$ is dense in $L^1(\mathbb{R})$.

has no real zero.

Then we set $\omega_\varepsilon(t) := \varepsilon \omega(\varepsilon t)$,

$W_\pm^{\text{Abel}}(H, H_0) \in \mathcal{B}(\mathcal{H})$ are defined by

the following relation (if it exist)

$$\lim_{\varepsilon \searrow 0} \int_0^\infty \omega_\varepsilon(t) \| e^{itH} e^{-itH_0} E_{ac}^{H_0} f - W_+^{\text{Abel}}(H, H_0) f \| dt^p = 0$$

for all $f \in \mathcal{H}$,

⚠ Tauberian theorem: $W_\pm^{\text{Abel}}(H, H_0)$

does not depend on ω and on $p \geq 1$.

[Yaf p76, BW Chap 6 p 97].

We could also consider $W_\pm^{\text{weak Abel}}(H, H_0)$

given by

$$w\text{-}\lim_{\varepsilon \searrow 0} \int_0^\infty \omega_\varepsilon(t) E_{ac}(H) e^{itH} E_{ac}(H_0) dt \quad (*)$$

⚠ These formulations are weaker than the original one, which means that the existence of these operators does NOT imply the existence of $W_{\pm}(H, H_0)$.

Let us consider $f_0 \in \mathcal{H}_{\text{pac}}(H_0)$ and $f \in \mathcal{H}_{\text{pac}}(H)$ and choose $w(t) = 2e^{-2t}$

$$\langle f, *f_0 \rangle$$

$$= \lim_{\varepsilon \searrow 0} \int_0^{\infty} 2\varepsilon e^{-2\varepsilon t} \langle e^{-itH} f, e^{-itH_0} f_0 \rangle dt$$

$$= \lim_{\varepsilon \searrow 0} 2\varepsilon \int_{-\infty}^{\infty} \langle \chi_{\mathbb{R}_+}(t) e^{-\varepsilon t} e^{-itH} f, \chi_{\mathbb{R}_+}(t) e^{-\varepsilon t} e^{-itH_0} f_0 \rangle dt$$

$$= \lim_{\varepsilon \searrow 0} \frac{2\varepsilon}{2\pi} \int_{\mathbb{R}} \left\langle \underbrace{(t \mapsto \chi_{\mathbb{R}_+}(t) e^{-\varepsilon t} e^{-itH} f)}_{\text{inverse Fourier}}, \underbrace{(t \mapsto \chi_{\mathbb{R}_+}(t) e^{-\varepsilon t} e^{-itH_0} f_0)}_{\text{inverse Fourier}} \right\rangle d\lambda$$

$$= \lim_{\varepsilon \searrow 0} \frac{\varepsilon}{\pi} \int_{\mathbb{R}} \langle (H - \lambda - i\varepsilon)^{-1} \varphi, (H_0 - \lambda - i\varepsilon)^{-1} \varphi_0 \rangle d\lambda$$