

III.2 Some resolvent equation

Question: Can one get H -smoothness from

H_0 -smoothness, by a perturbative argument?

For any self-adjoint op. $(H, D(H))$, we can consider

$D(H)$ with a new scalar product (and norm)

$$\langle f, g \rangle_{D(H)} := \langle f, g \rangle_{\mathcal{H}} + \langle Hf, Hg \rangle_{\mathcal{H}} \quad \forall f, g \in D(H)$$

$$\|f\|_{D(H)} = \sqrt{\langle f, f \rangle_{D(H)}} = \sqrt{\|f\|^2 + \|Hf\|^2}$$

It makes $D(H)$ a new Hilbert space.

We call $\|\cdot\|_{D(H)}$ the graph norm of H .

We shall often consider the situation

$$\langle Hf, g \rangle - \langle f, H_0 g \rangle = \langle Gf, G_0 g \rangle$$

$$\quad \forall f \in D(H) \quad \forall g \in D(H_0) \quad (*)$$

with $G_0 \in B(D(H_0), \mathcal{Y})$, $G \in B(D(H), \mathcal{Y})$

Remark Since $\|(H+i)f\|^2 = \langle (H+i)f, (H+i)g \rangle =$

$$= \|Hf\|^2 + \|f\|^2 + \underbrace{\langle Hf, if \rangle}_{=0} + \underbrace{\langle if, Hf \rangle}_{=0} = \|f\|_{D(H)}^2$$

Then $G \in B(D(H), Y)$

$$\Leftrightarrow \|Gf\|_Y^2 \leq c \|f\|_{D(H)}^2 = c \|(H+i)f\|^2$$

$\forall f \in D(H)$

Set $f = (H+i)^{-1}g$ for some $g \in Y_H$

$$\Leftrightarrow \|G(H+i)^{-1}g\|_Y^2 \leq c \|g\|_{Y_H}^2$$

$$\Leftrightarrow G(H+i)^{-1} \in B(Y_H, Y)$$

$$\Leftrightarrow G(H-z)^{-1} \in B(Y_H, Y) \text{ for any } z \in \rho(H)$$

Now, from \circledast , one has for $z \in \mathbb{C} \setminus \mathbb{R}$

$$\langle R(\bar{z})f, g \rangle - \langle f, R_0(z)g \rangle$$

$$= - \langle GR(\bar{z})f, G_0 R_0(z)g \rangle \quad \textcircled{O}$$

$$= - \langle f, (GR(\bar{z}))^* G_0 R_0(z)g \rangle$$

- 3 -

$$\Rightarrow R(z) - R_0(z) = - (GR(\bar{z}))^*(G_0 R_0(z))$$

Question

Can we write

$$R(z) - R_0(z) = - R(z) \underbrace{G^* G_0}_{\in B(Y, D(H)^*)} R_0(z) ?$$

$\in B(D(H)^*, Y)$ ↑
dual space

Without loss of too much generality, we can

consider $G_0 \in B(Y, Y)$ and $G \in B(Y, Y)$

We will assume this in the future

From now, we consider

$$R(z) - R_0(z) = - R(z) G^* G_0 R_0(z) \quad \begin{matrix} \text{multiply by} \\ \text{ } \\ \text{ } \end{matrix}$$

$\downarrow \quad \quad \quad G_0 \text{ and } G^*$

$$\Rightarrow G_0 R(z) G^* - G_0 R_0(z) G^* = -(G_0 R(z) G^*)(G_0 R_0(z) G^*)$$

$$\Rightarrow (\underbrace{1 - G_0 R(z) G^*}_{\text{ }})(\underbrace{1 + G_0 R_0(z) G^*}_{\text{ }}) = 1$$

Suppose that $G^* G_0$ is self-adjoint

$$\Rightarrow G^* G_0 = (G^* G_0)^* = G_0^* G$$

- 4 -

From ① with everything on the l.h.s., we get

$$R(z) - R_0(z) = -R_0(z) G_0^* G R(z)$$

$$= -R_0(z) G^* G_0 R(z)$$

multiply by G_0 and G^*

$$\Rightarrow G_0 R(z) G^* - G_0 R_0(z) G^* = -(G_0 R_0(z) G^*)(G_0 R(z) G^*)$$

$$\Rightarrow \underline{(1 + G_0 R_0(z) G^*)(1 - G_0 R(z) G^*)} = 1$$

Thus, we get $(1 - G_0 R(z) G^*)$ and $(1 + G_0 R_0(z) G^*)$

are boundedly invertible, and inverse of each others.

Then we have

$$R(z) = R_0(z) - \underbrace{R_0(z) - R_0(z) G^* G_0 R_0(z)}_{R(z) - R_0(z) G^* G_0 R(z)} \quad \text{□}$$

$$= R_0(z) - \underbrace{R_0(z) G^* G_0 R_0(z)}_{R_0(z) G^* G_0 R(z)} + \underbrace{R_0(z) G^* G_0 R(z) G^* G_0 R_0(z)}_{R_0(z) G^* G_0 R(z)}$$

$$= R_0(z) - \underbrace{R_0(z) G^*}_{R_0(z) G^*} (1 - G_0 R(z) G^*) G_0 R_0(z)$$

$$= R_0(z) - R_0(z) G^* (1 + G_0 R_0(z) G^*)^{-1} G_0 R_0(z)$$

does not contain $R(z)$!

Remark This equality is valid for any $z \in \rho(H) \cap \rho(H_0) \subset \mathbb{C} \setminus \mathbb{R}$

Let us still set

$$B_0(z) := -G_0 R_0(z) G^*$$

$$B(z) := G R(z) G^*$$

Then one has from \square

$$\underbrace{GR(z)G^*}_{B(z)} = GR_0(z)G^* - \underbrace{GR(z)G^*G_0R_0(z)G^*}_{B(z)}$$

$$\Rightarrow B(z)(1 + G_0 R_0(z) G^*) = GR_0(z) G^*$$

$$\Rightarrow \underline{B(z)} = GR_0(z) G^* (1 + G_0 R_0(z) G^*)^{-1}$$

$$= \underline{GR_0(z) G^* (1 - B_0(z))^{-1}}$$

III.3 Local existence and completeness of W_t

In the notation is introduced in the previous section

One has :

Thm 1. Let J be an open interval and suppose that

- 1) G_0 is H_0 -smooth on J

2) $GR_0(z)G^*$ is uniformly bounded for $\operatorname{Re}(z) \in J$
and $\operatorname{Im}(z) \neq 0$

3) $\sup_{\operatorname{Re}(z) \in J} \|G_0 R_0(z) G^*\| < 1$ *

Then $W_\pm(H, H_0, E^{H_0}(J))$ exist and are complete

(\Leftrightarrow existence of $W_\pm(H_0, H, E^H(J))$)

Remark * implies some smallness condition on

the perturbation G^*G_0 .

Proof

From the relation $B(z) = GR_0(z)G^*(1 - B_0(z))^{-1}$

together with the assumption 2) and 3)

We get that

$$\sup_{\substack{\operatorname{Re}(z) \in J \\ \operatorname{Im}(z) \neq 0}} \|B(z)\| = \sup_{\substack{\operatorname{Re}(z) \in J \\ \operatorname{Im}(z) \neq 0}} \|GR_0(z)G^*\| < \infty$$

$\Rightarrow G$ is H -smooth on J

With the assumption 1), we can use the main thm. of Section II.3 and get the result \square

Remark: We have seen that G is H -smooth on J

$$\text{if } \sup_{\substack{\lambda \in J \\ \varepsilon \neq 0}} \|G_J \operatorname{Im} R(\lambda + i\varepsilon) G_J^* \| < \infty$$

but this is still equivalent to

$$\sup_{\substack{\lambda \in J \\ \varepsilon \neq 0}} \|G \operatorname{Im} R(\lambda + i\varepsilon) G^* \| < \infty$$

In order to avoid the 3rd condition,

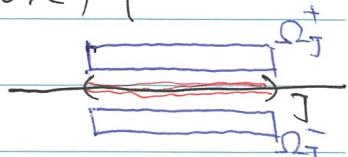
We can look more carefully at $(1 - B_0(z))^{-1}$

We need a preliminary result on holomorphic function.

Thm Let $J \subset \mathbb{R}$ an open interval and set

$$\Omega_J^\pm := \{z \in \mathbb{C} \mid \operatorname{Re}(z) \in J, \pm \operatorname{Im} z \in (0, \varepsilon)\}$$

Consider the map $\Omega_J^\pm \ni z \mapsto A(z) \in \mathbb{B}(y)$
s.t.



i) it is holomorphic

2) $A^\ell(z) \in K(\gamma)$ for some $\ell \in \mathbb{N}$ and all $z \in \Omega_J^\pm$

3) $\exists z_0 \in \Omega_J^\pm$ s.t. $(1 - A(z_0))^{-1} \in B(\gamma)$

4) the map admits a continuous extension to $J \subset \mathbb{C}$

Then, set $N := \{\lambda \in \bar{J} \mid 1 \in Q(A(\lambda \pm i0))\}$,

This set has Lebesgue measure 0 and is closed.

In addition, the map $z \mapsto (1 - A(z))^{-1}$ is meromorphic

on Ω_J^\pm and is continuous in norm as z approaches

a point in $\bar{J} \setminus N$.

[Yaf, p46-48]