and with $\| \varphi(A) \| \leq \| \varphi \|_\infty$.

[Amrein: Hilbert space §4.2].

II. Scattering theory: time dependent approach.

II.1) Evolution groups

Thm: $\exists$ bijective relation between self-adjoint operators $(A, \mathcal{D}(A))$ and strongly continuous unitary groups. (Stone's theorem)

A family $\{U_t\}_{t\in\mathbb{R}}$ is a strongly continuous unitary group if

1) U_t is unitary for any t,

2) $U_t U_s = U_{t+s}$ $\quad \forall t, s \in \mathbb{R}$,

3) $\operatorname*{s-lim}_{t\to 0} U_t = \mathbb{1}$.

Given $\{U_t\}_{t\in\mathbb{R}}$, one has

$$\mathcal{D}(A) := \left\{ f \in \mathcal{H} \;\middle|\; \underset{t\to 0}{\text{s-lim}} \frac{1}{t}(U_t - 1)f \text{ exist} \right\}$$

and then for $f \in \mathcal{D}(A)$, we set

$$Af := \underset{t\to 0}{\text{s-lim}} \frac{i}{t}(U_t - 1)f.$$

Formally, $U_t = \underbrace{e^{-itA} \quad \forall t \in \mathbb{R}}$.

this is well-defined by the functional calculus for $\varphi(\lambda) := e^{-it\lambda}$ but for each fixed $t \in \mathbb{R}$.

$\rightsquigarrow$ Check that $\{e^{-itA}\}_{t\in\mathbb{R}}$ is a strongly conti. unitary group.