

## V Schrödinger operators.

We illustrate previous concepts, but also use additional structures.

### V.1 The Laplace operator on $\mathbb{R}^d$ .

Consider  $\mathcal{H} = L^2(\mathbb{R}^d)$  and the Laplace operator

$$H_0 = -\Delta = -\sum_{j=1}^d \partial_j^2$$

and  $D(H_0) = \mathcal{H}^2(\mathbb{R}^d)$  ← Sobolev space of order 2

We set  $F: L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$  for

the Fourier transform with

$$[Ff](\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(x) dx$$

for  $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$  which extends to

a unitary operator.

Check that  $F H_0 F^* = X^2$  multiplication op

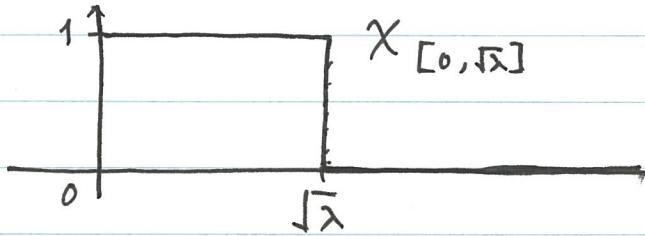
by the function  $\mathbb{R}^d \ni x \mapsto X^2 = \sum_{j=1}^d x_j^2 \in \mathbb{R}$ .

Then  $\mathcal{D}(H_0) = \mathcal{H}^2(\mathbb{R}^d) = F\mathcal{D}(X^2)$

with  $\mathcal{D}(X^2) = \{f \in L^2(\mathbb{R}^d) \mid x \mapsto x^2 f(x) \in L^2(\mathbb{R}^d)\}$

### Exercise

Check that  $E_\lambda^{H_0} = F X_{[0, \sqrt{\lambda}]} (|x|) F^*$



and that  $\mathcal{H}_{ac}(H_0) = \mathcal{H}$

$$\Rightarrow \mathcal{H}_{sc}(H_0) = \mathcal{H}_p(H_0) = \{0\}.$$

Consider  $\mathcal{H} = \int_{\mathbb{R}_+}^{\oplus} L^2(\mathbb{S}^{d-1}) d\lambda$

$$= L^2(\mathbb{R}_+, d\lambda; L^2(\mathbb{S}^{d-1}))$$
$$\cong L^2(\mathbb{R}_+, d\lambda) \otimes L^2(\mathbb{S}^{d-1}).$$

Then we set

$$F_0 : \mathcal{H} \rightarrow \mathcal{H}$$

$$[\mathcal{F}_0 f](\lambda, \omega) = [[\mathcal{F}_0 f](\lambda)](\omega)$$

$\in \mathbb{R}_+$        $\in \mathbb{S}^{d-1}$

$$:= 2^{-\frac{1}{2}} \lambda^{(d-2)/4} [Ff](\lambda^{\frac{1}{2}}\omega).$$

Check  $[\mathcal{F}_0 H_0 f](\lambda, \omega) = \lambda [\mathcal{F}_0 f](\lambda, \omega)$

$$\Leftrightarrow \mathcal{F}_0 H_0 \mathcal{F}_0^* = X \quad \text{multiplication by the function}$$

$$\mathbb{R}_+ \ni \lambda \mapsto \lambda \in \mathbb{R}$$

It means that  $(H_0, \mathcal{F}_0)$  determines a spectral representation of  $H_0$ .

Recall that  $\underline{F \varphi(H_0) F^* = \varphi(X^2)} \quad \forall \varphi \in L^\infty(\mathbb{R})$ .

$$\Rightarrow [(H_0 - z)^{-1} f](x) \Leftrightarrow \varphi(H_0) = F^* \varphi(X^2) F$$

$$= [F^* (X^2 - z)^{-1} F f](x)$$

$$= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{ix \cdot \xi} [(X^2 - z)^{-1} F f](\xi) d\xi$$

$$= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{ix \cdot \xi} (\xi^2 - z)^{-1} [F f](\xi) d\xi$$

$$\begin{array}{l} |x|^{\frac{d-2}{2}} \\ |x|^{d-1} \\ |x|^{d-3} \end{array}$$

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$$\begin{aligned} &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix \cdot \xi} (\xi^2 - z)^{-1} \left[ \int_{\mathbb{R}^d} e^{-i\xi \cdot x'} f(x') dx' \right] d\xi \\ &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left[ \int_{\mathbb{R}^d} e^{i\xi \cdot (x-x')} (\xi^2 - z)^{-1} dx' \right] f(x') dx' \\ &= \int_{\mathbb{R}^d} R_0(z; x, x') f(x') dx' \quad \text{for } f \in \mathcal{S}(\mathbb{R}^d) \end{aligned}$$

with  $R_0(z; x, x') = \frac{1}{(2\pi)^d} \int e^{i(x-x') \cdot \xi} (\xi^2 - z)^{-1} d\xi$ .

is called the Green function of  $H_0$

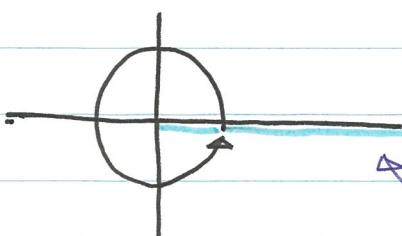
By computing this function in the sense of distributions, we get

$$R_0(z; x, x') = i \frac{1}{4} (2\pi)^{-\nu} z^{\nu/2} |x-x'|^{-\nu} H_\nu^{(1)}(z^{\nu/2} |x-x'|)$$

with  $\nu = \frac{d-2}{2}$

$\uparrow$   
Hankel function

with  $\arg(z) \in [0, 2\pi)$



cut in the complex plane

Note that for  $d=1$

$$R_0(z; x, x') = i 2^{-1} (2\pi)^{-1/2} \exp(i z^{1/2} |x-x'|)$$

and for  $d=3$

$$R_0(z; x, x') = (4\pi |x-x'|)^{-1} \exp(i z^{1/2} |x-x'|).$$

Observe that for  $\lambda > 0$  and  $\varepsilon > 0$ ,

one has  $(\lambda + i\varepsilon)^{1/2} = \lambda^{1/2} \left(1 + i\frac{\varepsilon}{\lambda}\right)^{1/2} \rightarrow \lambda^{1/2}$  as  $\varepsilon \rightarrow 0$

and  $(\lambda - i\varepsilon)^{1/2} = \lambda^{1/2} \left(\underbrace{1 - i\frac{\varepsilon}{\lambda}}_{\approx e^{2\pi i}}\right)^{1/2} \rightarrow \lambda^{1/2} e^{i\pi\nu}$ .

In particular for  $\nu = 1$

$$(\lambda + i\varepsilon)^{1/2} \longrightarrow \lambda^{1/2},$$

$$(\lambda - i\varepsilon)^{1/2} \longrightarrow -\lambda^{1/2}.$$

By using the relation

$$H_\nu^{(1)}(x) - e^{i\pi\nu} H_\nu^{(1)}(-x) = 2 I_\nu(x),$$

Bessel function

One has

$$\frac{1}{2\pi i} \left( R_o(\lambda + i\varepsilon; x, x') - R_o(\lambda - i\varepsilon; x, x') \right)$$

$$\begin{aligned} &\xrightarrow{\varepsilon \rightarrow 0} \frac{1}{2\pi i} i 4^{-1} (2\pi)^{-\nu} |x-x'|^{-\nu} \lambda^{\nu/2} \left\{ H_\nu^{(1)}(\lambda^{\nu/2}|x-x'|) - e^{i\pi\nu} H_\nu^{(1)}(-\lambda^{\nu/2}|x-x'|) \right\} \\ &= 2 I_\nu(\lambda^{\nu/2}|x-x'|) \\ &= \frac{1}{2} (2\pi)^{-\nu-1} \lambda^{\nu/2} |x-x'|^\nu I_\nu(\lambda^{\nu/2}|x-x'|). \quad (*) \end{aligned}$$



We have computed the limit  $\varepsilon \searrow 0$  of

$$\operatorname{Im} R_o(\lambda + i\varepsilon; x, x')$$
 for fixed  $x$  and  $x'$ .

It does not correspond to the convergence  
of  $G \operatorname{Im}(H_0 - \lambda - i\varepsilon)^{-1} G^*$  as  $\varepsilon \searrow 0$

for suitable  $G$ .

However, if the latter limit exists, then  $(*)$

will appear somewhere in its expression.

The unitary group associated with  $H_0$   
is also explicitly known:

$$[e^{-itH_0}f](x) = (4\pi it)^{-d/2} \int_{\mathbb{R}^d} \exp\left(\frac{i}{4t}|x-x'|^2\right) f(x') dx'$$

$$\text{with } (4\pi it)^{-d/2} = |4\pi t|^{-d/2} e^{-i\frac{\pi d}{4}} \text{ sgn}(t)$$

↑ sign functions.

Let us provide one more definition

which has no counterpart in the abstract framework:

$$\Gamma_0(\lambda) : \mathcal{S}(\mathbb{R}^d) \rightarrow L^2(\mathbb{S}^{d-1})$$

$$[\Gamma_0(\lambda)f](\omega) = [\mathcal{F}_0 f](\lambda, \omega).$$

$\Gamma_0(\lambda)$  extends then to a continuous

operator  $H^\alpha(\mathbb{R}^d)$  to  $L^2(\mathbb{S}^{d-1})$  for  $\alpha > \frac{1}{2}$

↑ Sobolev space of order  $\alpha$

$$= \{ f \in L^2(\mathbb{R}^d) \mid |x|^\alpha f \in L^2(\mathbb{R}^d) \}$$

$\Gamma_0(\lambda)$  is called a trace operator

or a restriction operator cf. Yaf II p71-75.

The relation with the abstract theory:

Since  $\langle X \rangle^{-\alpha} = \left\{ (1 + \sum_{j=1}^d x_j^2)^{\frac{1}{2}} \right\}^{-\alpha} \in \mathcal{B}(\mathbb{H})$

is locally  $H_0$ -smooth on  $(0, \infty)$  for

$\alpha > \frac{1}{2}$ ,  $Z(\lambda, \langle X \rangle^{-\alpha})$  is well-defined

and one has

$$Z(\lambda, \langle X \rangle^{-\alpha}) = \underline{\Gamma_0(\lambda)} \langle X \rangle^{-\alpha}$$

:  $\mathbb{H} \rightarrow L^2(S^{d-1})$  bounded.

## IV.2 The Schrödinger operator.

In  $\mathbb{H} = L^2(\mathbb{R}^d)$ , we shall consider

$H = H_0 + V$  with  $V$  a perturbation

of  $H_0$ . If  $V \in \mathcal{B}(\mathbb{H})$ , then

$D(H) = D(H_0)$ . If  $V$  is "small"

with respect to  $H_0$ , then we still have

$D(H) = D(H_0)$ .

But if  $V$  is not "small", then

$\mathcal{D}(H) \neq \mathcal{D}(H_0)$  and one has

to work for giving a meaning to  $H$ .

→ perturbation theory either

in the sense of operators or

in the sense of forms.

$$\begin{aligned} & \left( \text{look at } (\varphi, \psi) \mapsto \langle \varphi, H\psi \rangle \right) \\ & = \langle H^{k_2}\varphi, H^{k_2}\psi \rangle \\ & \text{if } H \geq 0 \end{aligned}$$