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Let \mathcal{H} be a Hilbert space, and let $(A, D(A))$ be a linear operator over \mathcal{H} . Let $\text{Op}(A)$ be the point spectrum of A .

- If A is self-adjoint and \mathcal{H} is separable, then functions in $\text{Op}(A)$ are orthogonal to each other, and $\text{Op}(A)$ is countable.

Proof. Let $f_1, f_2 \in \text{Op}(A)$ be eigenfunctions with respect to different eigenvalues ξ_1 and ξ_2 , respectively. Since A is self-adjoint, we have $\xi_1 \langle f_1, f_2 \rangle = \langle Af_1, f_2 \rangle = \langle f_1, Af_2 \rangle = \xi_2 \langle f_1, f_2 \rangle$, and $\langle f_1, f_2 \rangle = 0$ since $\xi_1 \neq \xi_2$. Then $\text{Op}(A)$ forms an orthogonal set in \mathcal{H} . But a separable Hilbert space can only have countable many orthogonal elements. Therefore, $\text{Op}(A)$ is countable. \square

- But if A is not self-adjoint, $\text{Op}(A)$ may not be countable.

For, let $\mathcal{H} = L^2([0, \infty)) = \{f \in C([0, \infty)) \mid \int_0^\infty |f|^2 dx < \infty\}$, and $(Af)(x) = f(x+1)$.

Claim: $\text{Op}(A)$ contains uncountably many eigenfunctions.

Proof. Put $f_\alpha := \alpha^x$ for $\alpha \in (0, 1)$. We have

$$\int_0^\infty |f_\alpha|^2 dx = \int_0^\infty \alpha^{2x} dx = \frac{\alpha^{2x}}{2 \ln \alpha} \Big|_0^\infty = \frac{1}{2 \ln \alpha} < \infty,$$

that is, $f_\alpha \in L^2([0, \infty))$. Note that $(Af_\alpha)(x) = \alpha^{x+1} = \alpha \cdot \alpha^x$.

We have $\{f_\alpha\}_{\alpha \in (0, 1)} \subset \text{Op}(A)$, but $\{f_\alpha\}$ contains uncountably many points. \square

In following context, we assume:

- \mathcal{H} = hilbert space
- all operators involved are bounded linear operators. $\mathcal{H} \rightarrow \mathcal{H}$.

Fact 1. If operator A is finite-dimensional, putting $A_R := A|_{R(A)}$ where $R(A)$ is the range of A , we have

$$(I - A)^{-1} = I + (I - A_R)^{-1}A$$

Proof. Since $[I + (I - A_R)^{-1}A](I - A) = (I - A) + (I - A_R)^{-1}(I - A)(A) = I - A + A = I$,

we have the desired equality holds. \square

Fact 2. For any bounded linear operator A , $l \in \mathbb{N}^+$,

$$\sigma(A^l) = \sigma(A)^l = \{\lambda \in \mathbb{C} \mid \lambda = \lambda^l \text{ for some } \lambda \in \sigma(A)\}.$$

Lemma 1. Let $A(\zeta) : \mathbb{C} \rightarrow \mathcal{B}(\mathcal{H})$ be an operator-valued function.

Suppose $A(\zeta)$ is holomorphic (in the weak topology) in a neighbourhood of ζ_0 and $1 \notin \sigma(A(\zeta_0))$. Then in a sufficiently small neighbourhood of ζ_0 , the operator $(I - A(\zeta))^{-1}$ exists, is bounded and depends holomorphically on ζ .

Proof. Since $F(\zeta_0) := (I - A(\zeta_0))^{-1}$ exists and bounded, we have

$$I - A(\zeta) = [I - (A(\zeta) - A(\zeta_0))F(\zeta_0)](I - A(\zeta_0)).$$

Choose r small enough such that $\|(A(\zeta) - A(\zeta_0))F(\zeta_0)\| < 1$ holds for $\zeta \in D_{\zeta_0}(r)$. By the lemma of Neumann Series, $I - (A(\zeta) - A(\zeta_0))F(\zeta_0)$ is invertible and we have

and holomorphic on ζ .

$$\begin{aligned} (I - A(\zeta))^{-1} &= (I - A(\zeta_0))^{-1} (I - (A(\zeta) - A(\zeta_0))F(\zeta_0))^{-1} \\ &= F(\zeta_0) (I - (A(\zeta) - A(\zeta_0))F(\zeta_0))^{-1} \end{aligned}$$

in $D_{\zeta_0}(r)$. \square

Theorem 1 (Analytic Fredholm alternative).

Suppose the operator-valued function $A(z)$ is holomorphic in Ω , $A^l(z)$ is a compact operator for some $l \in \mathbb{N}^+$, and for some $z_0 \in \Omega$ there exists the bounded operator $(I - A(z_0))^{-1}$. Then the operator-valued function $(I - A(z))^{-1} = F(z)$ is meromorphic with respect to $z \in \Omega$, and the set of its poles is

$$\mathcal{P} = \{z \in \Omega : 1 \in \sigma(A(z))\}.$$

Moreover, in the expansion of $F(z)$ in a Laurent series in a neighbourhood of any point $z_0 \in \mathcal{P}$, the coefficients of negative powers of $z - z_0$ are finite dimensional operators.

Proof. Applying Lemma 1, we find that $F(z)$ is holomorphic over $\Omega \setminus \mathcal{P}$.

Hence, it suffices to show that $\mathcal{P} \subset \{\text{poles of } f\}$, i.e.,

(1) \mathcal{P} is discrete;

(2) At each $z_0 \in \mathcal{P}$, f can be written as a Laurent series.

- Assume $A(z)$ is compact for any $z \in \Omega$.
We verify (1) first. Let $z_0 \in \Omega$.

Since $A(z_0)$ is compact, it is ~~the~~ limit of a sequence of finite-dim. op., i.e., we may write

$$A(z_0) = K_0 + B_0 \text{ where } k_0 = \dim R(K_0) < \infty$$

and $\|B_0\| = c_1 < 1$.

Choose $r = r(z_0)$ small enough, such that on the disk $D_{z_0}(r) = \{z | |z - z_0| < r\} \cap \Omega$, we have $\|A(z) - A(z_0)\| \leq C$ where $C = c_2 + c_3 < 1$.

Therefore, if we put $B(\zeta) := B_0 + A(\zeta) - A_{\zeta_0}$, we have

$$\|B(\zeta)\| \leq \|B_0\| + \|A(\zeta) - A(\zeta_0)\| \leq C_1 + C_2 < 1, \text{ and}$$

using the Lemma of Neumann series, we have ~~bounded~~

$(I - B(\zeta))^{-1}$ exists and ^{isomorphic} ~~bounded~~ on $D_{\zeta_0}(r)$.

Moreover, we have $\underbrace{(I - K(\zeta))(I - B(\zeta))}_{= I - A(\zeta)}$ if we put $K(\zeta) = K_0(I - B(\zeta))^{-1}$.

$K(\zeta)$ is holomorphic since $(I - B(\zeta))^{-1}$ is and. K_0 is finite-dimensional, ~~too~~ (And $K(\zeta)$ is also finite-dimensional).

Hence, over $D_{\zeta_0}(r)$, $(I - A(\zeta))^{-1}$ is not well-defined if and only if $I - K(\zeta)$ is not invertible, i.e., $\det(I - K(\zeta)) = 0$.

On the other hand, since $\det(I - K(\zeta))$ is a holomorphic function in respect of ζ , its zeros: either

- Admits no accumulation point in $D_{\zeta_0}(r)$
- or
- is the whole disk $D_{\zeta_0}(r)$.

However, we can exclude the second situation by the condition that

$(I - A(\zeta))^{-1}$ is bounded. Indeed, assuming $\zeta_0 \in \Omega$ is an accumulation

point of γ , Since Ω is connected, there is a continuous curve

$c: [0, 1] \rightarrow \Omega$ s.t. $c(0) = \zeta_0$ and $c(1) = \zeta_1$. Since c

is compact, there is a finite set of points: $\{\tilde{\zeta}_0, \dots, \tilde{\zeta}_p\}$ with $\tilde{\zeta}_0 = \zeta_0$ and $\tilde{\zeta}_p = \zeta_1$, such that

the corresponding disks: $D_{\tilde{z}_n}(r_n)$, $r_n = r_n(\tilde{z}_n)$ intersect pairwise.

Since $D_{\tilde{z}_p}(r_p) \subset \gamma$, we see that all the disks $D_{\tilde{z}_i}(r_i)$ ($i=0, \dots, p$) are contained in γ . However, $\tilde{z}_0 \notin \gamma$ as the condition suggests.

Which implies that γ is discrete.

Now let us verify (2). Since $I - A(\tilde{z}) = (I - K(\tilde{z})) (I - B(\tilde{z}))$, and $(I - B(\tilde{z}))^{-1}$ is holomorphic near $\tilde{z}_0 \in \gamma$, \tilde{z}_0 is a pole of $F(\tilde{z})$ iff it is a pole of $(I - K(\tilde{z}))^{-1}$. To study the power series expansion of $(I - K(\tilde{z}))^{-1}$, we note that $R(K(\tilde{z})) \subset R(K_0)$ and put ~~$K(\tilde{z}) = K_0 + \tilde{K}(\tilde{z})$~~ $\tilde{K} := (I - K(\tilde{z}))|_{R(K_0)}$.

Since $\dim R(K_0) = k_0$, $\det(\tilde{K}(\tilde{z}))$ have no zeros of order higher than k_0 . Using some linear algebra, we have.

$$\tilde{K}^{-1}(\tilde{z}) = I + \sum_{n=-n_0}^{\infty} A_n (\tilde{z} - \tilde{z}_n)^n$$

where $n_0 \leq k_0$ and $\dim A_0 \leq k_0$.

Using fact 1, we claim that $(I - K(\tilde{z}))^{-1}$ has a Laurent series at \tilde{z}_0 . This finishes the proof.

Now we consider the general case where $A(\tilde{z})$ doesn't have to be compact.

Suppose $A^k(\tilde{z}_0)$ is compact, we're going to show that there exist ~~on~~ $m \geq 1$ such that $\gamma \subset \widetilde{\gamma}$.

Since $A^k(\tilde{z}_0)$ is compact, (I referred to Wikipedia for this property).

$A^k(\tilde{z}_0)$ consists of eigenvalues accumulating only at 0.

By fact 2, $(\sigma(A(z_1))^l = \sigma(A^l(z_2))$, we claim that a

$A(z_2)$ ~~also~~ consists of eigenvalues accumulating only at zero.

(If not, suppose $\{z_n\} \subset \sigma(A(z_2))$ with $\lim_{n \rightarrow \infty} z_n = z_0 \neq 0$, then

$\{z_n^l\} \subset \sigma(A^l(z_2))$ with $z_n^l \rightarrow z_0^l$ as $n \rightarrow \infty$.)

The on the unit circle S^1 , there are only a finite number of

eigenvalues v_1, \dots, v_s of $A(z_2)$ (otherwise since S^1 is compact, there

will exists an accumulation point of $\sigma(A(z_2))$ on S^1). Then

$\sigma(A^m(z_2)) \cap S^1 = \{v_1^m, \dots, v_s^m\}$. Since $v_i \neq 1$ for $i=1, \dots, s$,

we may choose m large enough s.t. $v_i^m \neq 1$, $i=1, \dots, s$.

that is, $\eta \subset \widetilde{\eta}$. if we put $\widetilde{A}(z) = A^m(z)$ and

$$\widetilde{\eta} = \{z \in \Omega \mid z \in \sigma(\widetilde{A}(z))\}.$$

Using the argument above for compact operators, $\widetilde{\eta}$ is discrete,

then η is discrete. It suffices then to ~~show~~ verify the Laurent

series part.

Using the factorisation $I - A^m(z) = (I - A(z)) \cdot (I + A(z) + \dots + A^{m-1}(z))$,

we have the following equality:

$$(I - A(z))^{-1} = (I + A(z) + \dots + A^{m-1}(z)) (I - \widetilde{A}(z))^{-1},$$

and the Laurent series of the left hand's side can be derived
from that of ^{the} right hand side. \square

Theorem 2 Suppose λ is an interval of the real axis and the operator-valued function $A(z)$ satisfies the condition of Theorem 1 in the rectangle

$$\Omega = \Omega^{(\pm)} = \{z \in \mathbb{C} \mid \operatorname{Re} z \in \lambda, \pm \operatorname{Im} z \in D_0, E_0\}.$$

Suppose, moreover, that $A(z)$ is continuous in norm up to $\bar{\lambda}$. Then the set $N = N^{(\pm)} = \{z \in \bar{\lambda} \mid z \in \Omega(A(\lambda \pm i0))\}$

is closed and has Lebesgue measure 0, while the operator-valued function $F(z) = (I - A(z))^{-1}$ is meromorphic in Ω and is continuous in norm as z approaches points of the set $\bar{\lambda} \setminus N$.

Proof. From Theorem 2, $F(z)$ is meromorphic in Ω .

We are going to show that $F(z)$ is continuous in norm as $z \rightarrow \lambda \setminus N$. The argument is similar to that of Lemma 1:

If we put $F(z) := (I - A(z))^{-1}$, by definition for $z_0 = \lambda_0 \pm i0$, $F(z_0)$ exists and bounded. We have

$$I - A(z) = [I - (A(z) - A(z_0))F(z_0)](I - A(z_0)).$$

Choose r small enough, so that in disc $D_{z_0}(r)$, $\|(A(z) - A(z_0))F(z_0)\| < 1$. Then $I - (A(z) - A(z_0))F(z_0)$ is invertible and we have
and continuous

$$F(z) := (I - A(z))^{-1} = F(z_0)(I - (A(z) - A(z_0))F(z_0))^{-1} \text{ in } D_{z_0}(r),$$

then it follows that $F(z)$ continuous depends on z .

From the above fact, we claim that poles of $F(z)$ cannot accumulate at points of $\lambda \setminus N$. Otherwise for any $r > 0$, if $z_0 \in \lambda \setminus N$ is such a point, in disk $D_r(z_0)$ there would exist some poles of $F(z)$ which is unbounded. This is impossible since F is continuous at z_0 .

That is to say $\lambda \setminus N$ is open in λ , i.e., N is closed in λ .

Now it suffices to show that N has Lebesgue measure 0.

Again, we assume first $A(z)$ is compact for all $z \in \Omega$.

Since a compact operator is the limit of a sequence of finite rank operators, it's easy to see $A(\bar{z})$ is also ~~not~~ compact for \bar{z} in $\bar{\Lambda}$.

In this case, similar argument to that of Proof of theorem 2 can be applied. That is, we can write $A(\bar{z}) = k_0 + B_0$ where $k_0 = \dim R(k_0) < \infty$ and $\|B_0\| = c_1 < 1$, for all $\bar{z} \in \bar{\Lambda}$, and in a disk $D_{k_0}(r)$, we have

$$I - A(\bar{z}) = I - B(\bar{z}) - k_0 = (I - K(\bar{z})) (I - B(\bar{z}))$$

and

$$K(\bar{z}) = k_0 (I - B(\bar{z}))^{-1}$$

for $B(\bar{z}) = B_0 + A(\bar{z}) - A(\bar{z}_0)$.

In this case, in $D_{k_0}(r) \cap \Omega$ we have that

$$1 \in \sigma(A(\bar{z})) \text{ iff } \det(I - K(\bar{z})) = 0.$$

Argument for Theorem 2 showed that $\det(I - K(\bar{z}))$ is not identically zero in $D_{k_0}(r) \cap \Omega$.

By Luzin-Painlevé theorem, its zeros on $(-r, r)$ has measure zero. Since $\bar{\Lambda}$ is compact, we may repeat above process finitely many times to cover all of $\bar{\Lambda}$, and conclude that $|N|=0$ since $\bar{z} \in N$ is a ~~not~~ zero of $\det(I - K(\bar{z}))$.

Then we generate the result to the general case $A^l(\bar{z})$ is compact for some $l \geq 1$. Put $\widehat{A}(\bar{z}) = A^m(\bar{z})$, where $A^m(\bar{z})$ is such that

$$\{\bar{z} \in \Omega \mid 1 \in \sigma(A(\bar{z}))\} \subset \widehat{\Lambda} = \{\bar{z} \in \Omega \mid 1 \in \sigma(A^m(\bar{z}))\}$$

for some $m \geq l$. Then since the latter has 0-measure, the former also does. □