LIU Shuheng

Lemma. Let $\mathcal{H}$ be a Hilbert space, and let $\{f_n\} \subset \mathcal{H}$, $f_\infty \in \mathcal{H}$. Then $\{f_n\}$ converges strongly to $f_\infty \in \mathcal{H}$ if and only if that $\{f_n\}$ converges weakly to f_∞ with that $\lim_{n\to\infty} \|f_n\| = \|f_\infty\|$.

Proof. Let $s\text{-}\lim_{n\to\infty} f_n = f_\infty$. It fallows from the Cauchy-Schwarz inequality that $|\langle g, f_n - f_\infty\rangle|^2 \leq \|g\|^2 \cdot \|f_n - f_\infty\|^2$ for any $g \in \mathcal{H}$, and as $n \to \infty$ we have $\langle g, f_n - f_\infty\rangle$ goes to 0 since $\|f_n - f_\infty\|^2 \to 0$ as $\{f_n\}$ converages strongly. On the other hand, it fallows from triangle inequality that $|\|f_n\| - \|f_\infty\|| \leq \|f_n - f_\infty\|$ and by similar argument we get that $\{\|f_n\|\}$ converges to $\|f_\infty\|$ as real numbers.

Assume that $\{f_n\}$ converages weakly to f_∞ and $\lim_{n\to\infty} \|f_n\| = \|f_\infty\|$.

Note that $\langle f_n - f_\infty, f_n - f_\infty\rangle$

$$= \|f_n\|^2 + \|f_\infty\|^2 - \langle f_n, f_\infty\rangle - \langle f_\infty, f_n\rangle$$

$$= \|f_n\|^2 + \|f_\infty\|^2 - (\langle f_n, f_\infty\rangle - \langle f_\infty, f_\infty\rangle) - 2\langle f_\infty, f_\infty\rangle$$

$$= \|f_n\|^2 - \|f_\infty\|^2 - 2\mathrm{Re}\langle f_n - f_\infty, f_\infty\rangle$$

We have $\|f_n - f_\infty\|$ goes to 0 as n goes to ∞ since both $\|f_n\|^2 - \|f_\infty\|^2$ and $\langle f_n - f_\infty, -f_\infty\rangle$ has zero as their limits.

$\square$

名古屋大学大学院多元数理科学研究科