Report on Coulomb Scattering

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Abstract

Amrein section 5.8. (scattering theory for Coulomb potential) is summarized.

1 Modified wave operator

1.1 Introduction

For a Coulomb potential $V(\vec{x}) = \gamma |\vec{x}|^{-1}$ ($\vec{x} \in \mathbb{R}^n$, $n \geq 3$), the wave operator does not exist. So a more generalized version, namely the modified wave operator is introduced as the strong limit as $t \to \pm\infty$ of $U_t^* U_t^0 e^{-iX_t}$, where $\{X_t\}_{t\in\mathbb{R}}$ is a family of self adjoint operators commuting with $\{U_t^0\}_{t\in\mathbb{R}}$.

If we take $\{X_t\}_{t\in\mathbb{R}}$ such that

$$\frac{d}{dt} X_t\left(\vec{P}\right) = V\left(2\vec{P}t\right) \tag{1.1}$$

then it is a suitable choice in the sense that the existence of the strong limit is guaranteed.

Similar argument to Cook's criterion as below helps us to find a sufficient condition for the convergence.

$$U_s^* U_s^0 e^{-iX_s} f - U_t^* U_t^0 e^{-iX_t} f = \int_t^s \frac{\partial}{\partial \tau} U_\tau^* U_\tau^0 e^{-iX_\tau} f \tag{1.2}$$

$$\frac{d}{dt} U_t^* U_t^0 e^{-iX_t} f = i U_t^* \left[V - \frac{d}{dt} X_t\right] U_t^0 e^{-iX_t} f. \tag{1.3}$$

To show that the above choice is indeed suitable, it is enough to prove that the function $\left\| \left[V - \frac{d}{dt} X_t\right] U_t^0 e^{-iX_t} f \right\|$ of t is in $L^1([M, \infty))$.

1.2 Properties

For the modified wave operator $\Omega_\pm = \text{s}-\lim_{t\to\pm\infty} U_t^* U_t^0 e^{-iX_t}$, intertwining property

$$U_t \Omega_\pm = \Omega_\pm U_t^0 \tag{1.4}$$

holds, which is a consequence of the "feebly oscillating" property.

The modified wave operator is not uniquely determined. If X_t gives a modified wave operator, then $X_t + Z_t$ also gives another one, where $\{Z_t\}_{t\in\mathbb{R}}$ is a family of operators which are function of H_0 with $Z = \text{s}-\lim_{t\to\pm\infty} Z_t$ a self adjoint operator. The scattering operator S is unique up to unitary factor e^{iZ}, and the scattering matrix is unique up to $e^{i\alpha(\lambda)}$ (α is real).

1.3 The proof of existence of modified wave operator

For Coulomb potentials $V(\vec{x}) = \gamma |\vec{x}|^{-1}$ in $L^2(\mathbb{R}^n)$ with $n \geq 3$, the existence of Ω_+ is proven with $X_t\left(\vec{k}\right) = \gamma (2k)^{-1} \log t$, which satisfies (1.1).

The integrability of $\left\| \left[V - \frac{d}{dt}X_t\right] U_t^0 e^{-iX_t} f \right\|$ as the function of t for some neighborhood of ∞ is shown for $f \in D$, where the dense subset D of L^2 is defined as $\left\{ f \in L^2; \tilde{f} \in C_0^\infty \left(\mathbb{R}^n \backslash \{0\}\right) \right\}$.

For the proof, the potential $V(\vec{x})$ is decomposed into two parts as

$$V(\vec{x}) = \mathcal{V}(\vec{x}) + W(\vec{x}) \tag{1.5}$$

$W(\vec{x})$ is a short range part with bounded support and $\mathcal{V}(\vec{x})$ is a bounded long range part with the conditions

$$|\nabla \mathcal{V}(\vec{x})| \leq c\,(1 + |\vec{x}|)^{-2} \tag{1.6}$$

$$|\Delta \mathcal{V}(\vec{x})| \leq c\,(1 + |\vec{x}|)^{-3}. \tag{1.7}$$

$\left\| \left[\mathcal{V}\left(\vec{Q}\right) - \mathcal{V}\left(2\vec{P}t\right)\right] U_t^0 e^{-iX_t} f \right\|$ is shown to belong to $L^1([e,\infty))$, which is a main part of the proof of the integrability. The estimate below is derived for this.

$$\begin{aligned} &\left\| \left[\mathcal{V}\left(\vec{Q}\right) - \mathcal{V}\left(2\vec{P}t\right)\right] U_t^0 e^{-iX_t} f \right\| \\ &\leq\ c \int_0^1 d\epsilon \left[t \left\| \langle \epsilon Q \rangle^{-3} U_{t/\epsilon}^0 e^{-iX_t} f \right\| + \sum_{j=1}^n t \left\| \langle \epsilon Q \rangle^{-2} U_{t/\epsilon}^0 Q_j e^{-iX_t} f \right\| \right] \end{aligned} \tag{1.8}$$

where $\langle \epsilon Q \rangle = \left(I + \epsilon^2 \vec{Q}^2 \right)^{1/2}$. Then, the right hand side is shown to be in $L^1([e, \infty))$ from the inequalities

$$\left\| \langle \epsilon Q \rangle^{-k} U^0_{t/\epsilon} e^{-iX_t} f \right\| \leq c_k(f) (\log t)^k t^{-k} \tag{1.9}$$

$$\left\| \langle \epsilon Q \rangle^{-k} U^0_{t/\epsilon} Q_j e^{-iX_t} f \right\| \leq \tilde{c}_k(f) (\log t)^{k+1} t^{-k} \tag{1.10}$$

where $k = \mathbb{N} \cup \{0\}$. These estimates themselves are proven by induction. The key idea is to reduce the degree of the term $\langle \epsilon Q \rangle^{-k}$ by using the equalities:

$$U^0_{t/\epsilon} Q_j e^{-iX_t} f = \frac{\epsilon}{2t} \vec{Q} \cdot \vec{P} H_0^{-1} U^0_{t/\epsilon} e^{-iX_t} f - \frac{\epsilon}{2t} U^0_{t/\epsilon} \vec{Q} \cdot \vec{P} H_0^{-1} e^{-iX_t} f \tag{1.11}$$

$$\left\| \langle \epsilon Q \rangle^{-1} \epsilon Q_j \right\| = 1. \tag{1.12}$$

2 Nonexistence of wave operator

Standard wave operator does not exist in the case of Coulomb potential system. This is proven by contradiction (for $t \to \infty$).

Assume that it exists and denote it W_+, so $W_+ = \text{s-}\lim_{t \to \pm\infty} U_t^* U_t^0$. Then the following equation gives contradiction:

$$\text{s-}\lim_{t \to \infty} \left[\Omega_+ e^{iX_t} - U_t^* U_t^0 \right] = O. \tag{2.1}$$

This leads to

$$\text{w-}\lim_{t \to \infty} U_t^* U_t^0 = O. \tag{2.2}$$

But $W_+ = \text{w-}\lim_{t \to \infty} U_t^* U_t^0$, and W_+ must be an isometric operator, so there is a contradiction.

The first equation is due to the estimate

$$\left\| \Omega_+ e^{-iX_t} f - U_t^* U_t^0 e^{-iX_t} f \right\| \leq \int_t^\infty \left\| \left[V\left(\vec{Q}\right) - V\left(2\vec{P}t\right) \right] U_s^0 e^{-iX_s} e^{-iX_t} f \right\| ds \tag{2.3}$$

and the fact that the integrand of the right hand side is majorized by a t-independent function in $L^1([e, \infty))$, which follows from similar inequalities to (1.9) and (1.10), leads to (2.1).

The second equation is due to (2.1) and

$$\left\langle g, U_t^* U_t^0 f \right\rangle = \left\langle \Omega_+^* g, e^{iX_t} f \right\rangle - \left\langle g, \left[\Omega_+ e^{iX_t} - U_t^* U_t^0 \right] f \right\rangle \tag{2.4}$$

and

$$\text{w-}\lim_{t\to\infty} e^{iX_t} = O. \tag{2.5}$$

(2.5) is obtained from the equation

$$\left\langle g, e^{iX_t} f \right\rangle = \int_0^\infty e^{i\gamma(\log t)/(2k)} \theta(k)\, dk \tag{2.6}$$

for $f, g \in D$, where $\theta(k) = k^{n-1} \int_{S^{n-1}} h(k\vec{\omega})\, d\omega$, and Riemann-Lebesgue Lemma.