

Notation

\mathcal{H} : Hilbert space. Assume it is separable.

A : a self-adjoint operator in \mathcal{H} . ($A: \mathcal{H} \rightarrow \mathcal{H}$)

$E_\lambda = E_\lambda^A$: the spectral family of A .

$A_B := \{A \subset \mathbb{R}; A \text{ is a Borel set}\}$.

$E(V)$: the spectral measure corresponding to E_λ . $V \in A_B$.

$\mathcal{M}(\{x\}) := \{f \in \mathcal{H}; E(\{x\})f = f\}$

$m_f(V)$: a measure on the measure space (\mathbb{R}, A_B) defined by

$$m_f(V) := \langle f, E(V)f \rangle = \|E(V)f\|^2.$$

$F_f(\lambda)$: the mapping (function) $\lambda \mapsto \|E_\lambda f\|^2$.

Note that $m_f(V)$ is the Stieltjes measure associated with F_f .

$\sigma_p(A) := \{\text{eigenvalue of } A\} = \{\lambda \in \mathbb{R}; \exists f \in \mathcal{H}, f \neq 0 \text{ s.t. } Af = \lambda f\}$

As \mathcal{H} is separable now, $\#\sigma_p(A)$ is at most countable.

$$\text{supp } \{E_\lambda\} = \{\mu \in \mathbb{R}; E_{\mu+\epsilon} - E_{\mu-\epsilon} \neq 0, \forall \epsilon > 0\}.$$

We say that E_λ is supported on this set.

$$\text{supp } \{F_f\} := \{\mu \in \mathbb{R}; F_f(\mu+\epsilon) - F_f(\mu-\epsilon) \neq 0, \forall \epsilon > 0\}.$$

We say that m_f is supported on this set.

Def 1.

$$\mathcal{H}_p(A) := \bigoplus_{\lambda \in \text{sp}(A)} \mathcal{M}(\lambda)$$

Remark

$$\mathcal{H}_p(A) = \overline{\text{span} \bigcup_{\lambda \in \text{sp}(A)} \mathcal{M}(\lambda)}$$

Def 2.

$$\mathcal{H}_c(A) := \{f \in \mathcal{H}; m_f \text{ is a continuous measure.}\}$$

$$= \{f \in \mathcal{H}; F_f \text{ is a continuous function.}\}$$

Remark

m_f is a continuous measure

def

$$\Leftrightarrow m_f(\{\lambda\}) = 0, \forall \lambda \in \mathbb{R}.$$

We try to prove the following theorem

Thm $\mathcal{H} = \mathcal{H}_p(A) \oplus \mathcal{H}_c(A).$

We use the following lemmas.

Lem 1. $f \in \mathcal{M}(\{\mu\}) \iff Af = \mu f.$

Lem 2. $\mu \notin \sigma_p(A) \implies E(\{\mu\})f = 0.$

proof of Lem 1. (Note that if $f=0$ it is trivial. We prove the case $f \neq 0$.)

We use the formula (eq. (4.42) of [Amrein]) with $a=-\infty, L=\infty, \varphi(\lambda) = \lambda - \mu$:

$$\left\| \int_a^L \varphi(\lambda) E(d\lambda) f \right\|^2 = \int_a^L |\varphi(\lambda)|^2 m_+(d\lambda),$$

and functional calculus.

$$\|Af - \mu f\|^2 = \left\| \int_{-\infty}^{\infty} (\lambda - \mu) E(d\lambda) f \right\|^2 = \int_{-\infty}^{\infty} |\lambda - \mu|^2 m_+(d\lambda).$$

(\Rightarrow) Assume $f \in \mathcal{M}(\{\mu\})$. Then, $E(\{\mu\})f = f$. Furthermore, if $V \subset \{\mu\}^c$,

then $E(V)f = 0$. $\therefore m_+(V) = 0$ ($V \subset \{\mu\}^c$).

$\therefore \|Af - \mu f\|^2 = 0$ and $Af = \mu f$.

(\Leftarrow) Assume $Af = \mu f$. Then $m_{\mu}(V) = 0$ ($V \subset \{\mu\}^{\circ}$).

This means $E(V)f = 0$ ($\forall V \subset \{\mu\}^{\circ}$).

$\therefore E(\{\mu\})f = f$. $\therefore f \in M(\{\mu\})$. \square

proof of Lem 2.)

Assume $\mu \in \sigma_p(A)$. Then, $Af = \mu f$ ($\forall f \in \mathcal{H}, f \neq 0$).

From Lem 1., $f \in M(\{\mu\})$ ($\forall f \in \mathcal{H}, f \neq 0$).

As $M(\{\mu\}) = \{f \in \mathcal{H}; E(\{\mu\})f = f\}$,

$E(\{\mu\})f \neq f, \forall f \in \mathcal{H}, f \neq 0$. ---- (*)

Put $g_f := E(\{\mu\})f$. Then, $E(\{\mu\})g_f = E(\{\mu\})^2 f = E(\{\mu\})f = g_f$.

If $\exists g_f \neq 0$, it contradicts with (*), so $g_f = 0$ ($\forall f \in \mathcal{H}$).

This means $E(\{\mu\})f = 0, \forall f \in \mathcal{H}$. \square

proof of the theorem

$\mathcal{H}_p(A)$ is a subspace of \mathcal{H} , as we can see from its definition.

Therefore, $\mathcal{H} = \mathcal{H}_p(A) \oplus \mathcal{H}_p(A)^\perp$ — $(*)$

We show that $\mathcal{H}_p(A)^\perp$ coincides with $\mathcal{H}_c(A)$.

$$(\mathcal{H}_c(A) \subset \mathcal{H}_p(A)^\perp)$$

Let $f \in \mathcal{H}_c(A)$. Then, $E(i\lambda)f = 0$ ($\forall \lambda \in \mathbb{R}$).

Note that $P_{\mathcal{H}_p(A)} = \sum_{\lambda \in \sigma_p(A)} E(i\lambda)$. So, $P_{\mathcal{H}_p(A)}f = 0$. ($P_{\mathcal{H}_p(A)}$ is the projection operator on $\mathcal{H}_p(A)$.)

From $(*)$, we can decompose f , $f = f_1 + f_2$ ($f_1 \in \mathcal{H}_p(A)$, $f_2 \in \mathcal{H}_p(A)^\perp$).

As $f_1 = P_{\mathcal{H}_p(A)}f = 0$, $f \in \mathcal{H}_p(A)^\perp$.

$$(\mathcal{H}_c(A) \supset \mathcal{H}_p(A)^\perp)$$

Let $f \in \mathcal{H}_p(A)^\perp$. Then, $E(i\lambda)f = 0$ ($\forall \lambda \in \sigma_p(A)$).

From Lem. 2, we see that $E(i\lambda)f = 0$ for all $\lambda \in \mathbb{R}$.

This means $f \in \mathcal{H}_c(A)$.

□