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Topics in Mathematical Science II

Report

date: 2018/04/18

name: Tomotaka KUNISADA

student#: 321801071

e-mail: m18011a@math.nagoya-u.ac.jp

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Notation

\mathcal{H} Hilbert space on \mathbb{C} with scalar product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$.

• Strong Convergence

A sequence $\{f_n\}_{n \in \mathbb{N}} \subset \mathcal{H}$ is said to be strongly convergent to $f_\infty \in \mathcal{H}$ and written $s\text{-}\lim_{n \rightarrow \infty} f_n = f_\infty$

iff. $\lim_{n \rightarrow \infty} \|f_n - f_\infty\| = 0$.

• Weak Convergence

A sequence $\{f_n\}_{n \in \mathbb{N}} \subset \mathcal{H}$ is said to be weakly convergent to $f_\infty \in \mathcal{H}$ and written $w\text{-}\lim_{n \rightarrow \infty} f_n = f_\infty$

iff. $\lim_{n \rightarrow \infty} \langle g, f_n - f_\infty \rangle = 0, \forall g \in \mathcal{H}$.

Statements to be proven in this report

In this report, I will give proofs for two statements;

1. strongly convergence implies weak convergence.

2. the lemma introduced in the lecture (on the condition in which weak convergence coincides with strong convergence)

Precise form and proof of the above statements will be given in the following.

1. Let $\{f_n\}_{n \in \mathbb{N}} \subset \mathcal{H}$ strongly converge to $f_\infty \in \mathcal{H}$ i.e.,

$$s\text{-}\lim_{n \rightarrow \infty} f_n = f_\infty.$$

Then, $\{f_n\}_{n \in \mathbb{N}}$ weakly converges to f_∞ i.e.,

$$w\text{-}\lim_{n \rightarrow \infty} f_n = f_\infty.$$

proof) Schwarz inequality implies

$$|\langle g, f_n - f_\infty \rangle| \leq \|g\| \cdot \|f_n - f_\infty\|, \quad \forall g \in \mathcal{H}.$$

Since $s\text{-}\lim_{n \rightarrow \infty} f_n = f_\infty$, we have $\lim_{n \rightarrow \infty} \|f_n - f_\infty\| = 0$.

Therefore, as $0 \leq |\langle g, f_n - f_\infty \rangle|$,

$$\lim_{n \rightarrow \infty} |\langle g, f_n - f_\infty \rangle| = 0.$$

So,

$$\lim_{n \rightarrow \infty} \langle g, f_n - f_\infty \rangle = 0, \quad \forall g \in \mathcal{H}.$$

This is identical with the weak convergence, i.e.,

$$w\text{-}\lim_{n \rightarrow \infty} f_n = f_\infty.$$

□

2. Let $\{f_n\}_{n \in \mathbb{N}} \subset \mathcal{H}$ be a sequence in \mathcal{H} .

Then the following holds;

$\{f_n\}_{n \in \mathbb{N}}$ strongly converges to f_∞ i.e., $s\text{-}\lim_{n \rightarrow \infty} f_n = f_\infty$.

$\Leftrightarrow \{f_n\}_{n \in \mathbb{N}}$ weakly converges to f_∞ i.e., $w\text{-}\lim_{n \rightarrow \infty} f_n = f_\infty$,

and $\lim_{n \rightarrow \infty} \|f_n\| = \|f_\infty\|$.

proof) (\Rightarrow)

It remains to prove $\lim_{n \rightarrow \infty} \|f_n\| = \|f_\infty\|$.

First, $\lim_{n \rightarrow \infty} \langle f_\infty, f_n - f_\infty \rangle = 0$ is known from the weak convergence of $\{f_n\}_{n \in \mathbb{N}}$ to f_∞ .

$$\begin{aligned} \text{Then, } \lim_{n \rightarrow \infty} \langle f_n, f_n \rangle &= \lim_{n \rightarrow \infty} \langle f_n, f_\infty \rangle + \langle f_n, f_n - f_\infty \rangle \\ &= \lim_{n \rightarrow \infty} \langle f_n, f_\infty \rangle + \langle f_\infty, f_n - f_\infty \rangle + \langle f_n - f_\infty, f_n - f_\infty \rangle \\ &= \lim_{n \rightarrow \infty} \langle f_n, f_\infty \rangle + \langle f_\infty, f_n - f_\infty \rangle + \|f_n - f_\infty\|^2 \\ &= \lim_{n \rightarrow \infty} \langle f_\infty, f_\infty \rangle + \langle f_n - f_\infty, f_\infty \rangle + \langle f_\infty, f_n - f_\infty \rangle + \|f_n - f_\infty\|^2 \\ &= \lim_{n \rightarrow \infty} \|f_\infty\|^2 + \overline{\langle f_\infty, f_n - f_\infty \rangle} + \langle f_\infty, f_n - f_\infty \rangle + \|f_n - f_\infty\|^2; \end{aligned}$$

Since $s\text{-}\lim_{n \rightarrow \infty} f_n = f_\infty$, $\lim_{n \rightarrow \infty} \|f_n - f_\infty\| = 0$,

Therefore, $\lim_{n \rightarrow \infty} \langle f_n, f_n \rangle = \|f_\infty\|^2$, and then

$\langle f_n, f_n \rangle = \|f_n\|^2$ leads to the conclusion,

$$\lim_{n \rightarrow \infty} \|f_n\| = \|f_\infty\|.$$

(\Leftarrow)

$$\begin{aligned}
 \|f_n - f_\infty\|^2 &= \langle f_n - f_\infty, f_n - f_\infty \rangle \\
 &= \langle f_n - f_\infty, f_n \rangle - \langle f_n - f_\infty, f_\infty \rangle \\
 &= \langle f_n, f_n \rangle - \langle f_\infty, f_n \rangle - \langle f_n, f_\infty \rangle + \langle f_\infty, f_\infty \rangle \\
 &= \|f_n\|^2 - \langle f_\infty, f_n \rangle - \langle f_n, f_\infty \rangle + \|f_\infty\|^2
 \end{aligned}$$

Now, the weak convergence of $\{f_n\}_{n \in \mathbb{N}}$ to f_∞ implies

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \langle f_\infty, f_n \rangle &= \lim_{n \rightarrow \infty} \langle f_\infty, f_n - f_\infty \rangle + \langle f_\infty, f_\infty \rangle \\
 &= \langle f_\infty, f_\infty \rangle = \|f_\infty\|^2, \text{ and}
 \end{aligned}$$

$$\lim_{n \rightarrow \infty} \langle f_n, f_\infty \rangle = \lim_{n \rightarrow \infty} \overline{\langle f_\infty, f_n \rangle} = \overline{\|f_\infty\|^2} = \|f_\infty\|^2.$$

Then, if $\lim_{n \rightarrow \infty} \|f_n\| = \|f_\infty\|$,

$$\lim_{n \rightarrow \infty} \|f_n - f_\infty\|^2 = \|f_\infty\|^2 - \|f_\infty\|^2 - \|f_\infty\|^2 + \|f_\infty\|^2 = 0.$$

This means $s\text{-}\lim_{n \rightarrow \infty} f_n = f_\infty$.

Now the proof is finished. \square