A remark on point spectrum

If $(A, \mathcal{D}(A))$ is a self-adjoint operator in a separable Hilbertspace $\mathcal{H}$, then by orthogonality of eigenfunctions, $\sigma_p(H)$ is countable. (This is also true for normal op^s see Rudin's Functional analysis)

However, if A is not self-adjoint, then this is not always true.

Example Let $\mathcal{H} = \ell^2(\mathbb{N}) := \left\{ a = (a_n)_{n=0}^{\infty} \subset \mathbb{C} \;\middle|\; \sum_{n=0}^{\infty} |a_n|^2 < \infty \right\}$.

Consider the shift operator S defined by

$$(Sa)_n := a_{n+1} \quad \text{for each } n \in \mathbb{N}.$$

$$\begin{array}{cccccccc}
 & a_0 & a_1 & a_2 & a_3 & a_4 & \cdots\cdots & a_n & a_{n+1} \\
S \downarrow & & \swarrow & \swarrow & \swarrow & \swarrow & & & \swarrow \\
 & a_1 & a_2 & a_3 & a_4 & \cdots\cdots & & a_{n+1} & \\
 & \text{0-th} & \text{1st} & \text{2nd} & \text{3rd} & \cdots\cdots & & &
\end{array}$$

Take $\lambda \in \mathbb{C}$ with $|\lambda| < 1$ and set

$$a^{(\lambda)} = (a_n^{(\lambda)})_{n=0}^{\infty} = (1, \lambda, \lambda^2, \lambda^3, \cdots\cdots).$$

Then, since $\sum_{n=0}^{\infty} |a_n^{(\lambda)}|^2 = \sum_{n=0}^{\infty} |\lambda|^{2n} = \frac{1}{1-|\lambda|^2} < \infty$,

$a^{(\lambda)} \in \ell^2(\mathbb{N})$. Moreover, for $\forall n \in \mathbb{N}$

$$(Sa^{(\lambda)})_n = a_{n+1}^{(\lambda)} = \lambda^{n+1} = \lambda a_n^{(\lambda)}$$

hold. This means $a^{(\lambda)}$ is an eigenfunction of S with eigenvalue λ. Hence, $\sigma_p(S)$ is <u>uncountable</u>.

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