

A remark on point spectrum

If $(A, \mathcal{D}(A))$ is a self-adjoint operator

in a separable Hilbertspace \mathcal{H} , then

by orthogonality of eigenfunctions, $\sigma_p(H)$

is countable. (This is also true for normal ops
see Rudin's Functional analysis)

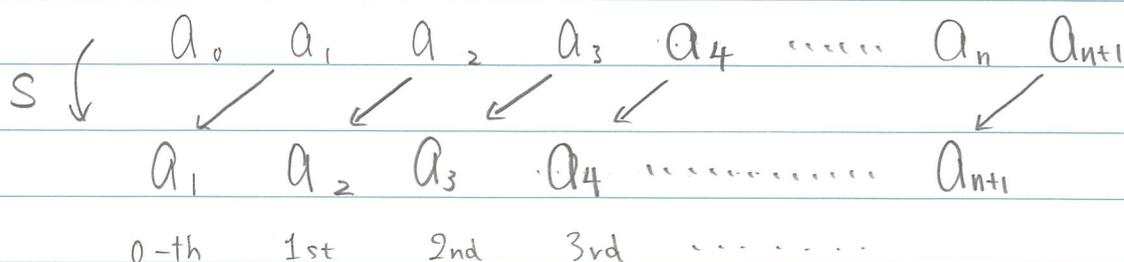
However, if A is not self-adjoint,

then this is not always true.

Example Let $\mathcal{H} = \ell^2(\mathbb{N}) := \left\{ a = (a_n)_{n=0}^{\infty} \in \mathbb{C} \mid \sum_{n=0}^{\infty} |a_n|^2 < \infty \right\}$.

Consider the shift operator S defined by

$$(Sa)_n := a_{n+1} \quad \text{for each } n \in \mathbb{N}.$$



Take $\lambda \in \mathbb{C}$ with $|\lambda| < 1$ and set

$$a^{(\lambda)} = (a_n^{(\lambda)})_{n=0}^{\infty} = (1, \lambda, \lambda^2, \lambda^3, \dots).$$

Then, since $\sum_{n=0}^{\infty} |a_n^{(\lambda)}|^2 = \sum_{n=0}^{\infty} |\lambda|^{2n} = \frac{1}{1-|\lambda|^2} < \infty$,

$a^{(\lambda)} \in \ell^2(\mathbb{N})$. Moreover, for $\forall n \in \mathbb{N}$

$$(Sa^{(\lambda)})_n = a_{n+1}^{(\lambda)} = \lambda^{n+1} = \lambda a_n^{(\lambda)}$$

hold. This means $a^{(\lambda)}$ is an eigenfunction

of S with eigenvalue λ . Hence, $\sigma_p(S)$

is uncountable.

(H. Inoue)