Functional Calculus of Unbounded Operator (revise at 10th August)

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1 Introduction

In some of exercises and scattering theory, functional calculus plays an essential role. That is because I proof functional calculus of unbounded operator version for my subject.

Functional calculus says that any self adjoint operator (even if on infinity dimension or unbounded) can calculus like complex function. Therefore we regard spectral decomposition as a kind of integration, represent unbounded operator (such as differential operator or Hamiltonian) as a unitary group which is well-known as Stone's theorem, and so on.

2 Proof of functional calculus

First, we check the notation. $\mathcal{H} \neq (0)$ is Hilbert space and $\mathcal{B}(\mathcal{H})$ is a bounded linear operators on $\mathcal{H}$. $\mathcal{C}(\mathcal{H})$ is densely defined closed linear operators on $\mathcal{H}$ which have a domain. $\sigma(S)$ is the spectral set of S and $\rho(S)$ is the resolvent set of S. If $\lambda \in \rho(S)$, then $\lambda - S$ has inverse and this inverse denoted by $(\lambda - S)^{-1}$ or R_λ. $\sigma(\tilde{S})$ is defined by onepoint conpactification when $\sigma(S)$ is unbounded, if σS is unbounded, $\sigma(\tilde{S})$ is defined by $\sigma(S)$. We write $e(s) \equiv 1$ and $r_\lambda(s) = \dfrac{1}{\lambda - s}$, $\lambda \in \rho(S)$ for some S.

At first, we show following proposition

Proposition 2.1. Let $S : \mathcal{H} \to \mathcal{H}$ be densely defined and symmetric. The following are eauivalent:

1. $S = S^*$
2. $\sigma(S) \subset \mathbb{R}$
3. $\pm i \in \rho(S)$.

Proof. Suppose 1 and suppose $\lambda = \mu + i\nu$ with μ, ν real and $\nu \neq 0$. When $x \in D(S)$, $|((\lambda - S)x, (\lambda - S)x)| \geq |\lambda|^2 ||x||^2 \geq |\nu|^2 ||x||^2$. Therefore $\lambda - S$ is 1-1 and has closed range sinse $S = S^*$ is closed. Again $\mathcal{R}(\lambda - S)^\perp = N((\lambda - S)^*) = N(\overline{\lambda} - S) = (0)$ with $N(S)$ be a kernel of S. Thus $\lambda - S$ is onto, so $\lambda \in \rho(S)$.

$2 \Rightarrow 3$ is clear. Suppose 3. Then $N(S^* + i) = \mathcal{R}(S^* - i)^\perp = (0)$. This means that $S^* + i$, which is an extension of the onto operator $S + i$, is 1-1. This is only possible if $S^* + i = S + i$, which implies $S^* = S$.

□

Next, we show the theorem which should be called the "continuous" functional calculus.

Theorem 2.2. Let $S : \mathcal{H} \to \mathcal{H}$ be a self adjoint operator. The map $e \mapsto I,\ r_\lambda \mapsto (\lambda - S)^{-1}$ has a unique extension and this extension is an isometric algebra isomorphism $f \mapsto f(S) : C(\tilde{\sigma}(S)) \to \mathcal{B}$. Moreover, this extension preserves adjoint and order, i.e. $f(S)^* = f^*(S)$, $f(S) \geq 0$ if and only if $f \geq 0$, on $\sigma(S)$.

Proof. Let $\mathcal{F}$ be the algebra of functions on $\mathbb{C}$ generated by e and r_λ. When such map exists and let p be a polynomial in n-variables, then the function f defined by $f(s) = p(r_{\lambda_1}(s), \cdots, r_{\lambda_n}(s))$ correspond to the operator $f(S) = p(R_{\lambda_1}, \cdots, R_{\lambda_n})$. Conversely, this will extend the map to $\mathcal{F}$ when it is well-defined, i.e. $p(r_{\lambda_1}(s), \cdots, r_{\lambda_n}(s)) = 0$ imply $p(R_{\lambda_1}, \cdots, R_{\lambda_n}) = 0$.

To show this, we induce on n. For $n = 1$, $p(r_{\lambda_1}) \equiv 0$ imply p is zero polynomial, so $p(R_{\lambda_1}) \equiv 0$. Next, suppose the assumption is true in $n \leq m$, and suppose $p(r_{\lambda_1}(s), \cdots, r_{\lambda_{m+1}}(s)) \equiv 0$. Since each $r_{\lambda_j} = 0$ at ∞, the constant term of p is zero.

Now,

$$\begin{aligned}(\lambda_{m+1} - s) r_{\lambda_j}(s) &= (\lambda_{m+1} - s)\frac{1}{\lambda_j - s} \\ &= \lambda_{m+1}\frac{1}{\lambda_j - s} + 1 - \lambda_j \frac{1}{\lambda_j - s} \\ &= (\lambda_{m+1} - \lambda_j) r_{\lambda_j} + 1,\end{aligned}$$

so $(\lambda_{m+1} - s) p(r_{\lambda_1}(s), \cdots, r_{\lambda_{m+1}}(s)) = q(r_{\lambda_1}(s), \cdots, r_{\lambda_m}(s))$, with q is polynomial in m-variables because p has no constant term. Similarly,

$$(\lambda_{m+1} - S) R_{\lambda_j} = (\lambda_{m+1} - \lambda_j) R_{\lambda_j} + I,$$

so $(\lambda_{m+1} - S) p(R_{\lambda_1}, \cdots, R_{\lambda_{m+1}}) = q(R_{\lambda_1}, \cdots, R_{\lambda_m})$ and this is equal to 0 by the assumption of induction. Finally, p has no constant terms, each R_{λ_j} has range $D(S)$, and $\lambda_{m+1} - S$is $1-1$, so these impies $p(R_{\lambda_1}, \cdots, R_{\lambda_{m+1}}) \equiv 0$. Thus there is a unique extension to an algebra homomorphism $f \mapsto f(S) : \mathcal{F} \to \mathcal{B}(\mathcal{H})$.

Next, we show that for $f \in \mathcal{F}$,$\sigma(f(S)) = f(\tilde{\sigma}(S))$. To show this, note that $\mathcal{F}$ consists precisely of all bounded rational functions which are holomorphic on $\sigma(S)$. Suppose $x \notin f(\tilde{\sigma}(S))$. Then put $(x - f(s))^{-1} = h(s) \in \mathcal{F}$ and by homomophismness,

$$(x - f(S))h(S) = e(S) = I = h(S)(x - f(S)).$$

Thus $x \notin \sigma(f(S))$. Conversely if $\lambda \in \sigma(S)$ and $x = f(\lambda)$, then $s - f(s) = (\lambda - s)g(s)$, where $g(s) \in \mathcal{F}$ and $g(\infty) = 0 by boundness$. Therefore, using the fractions decomposition of g we get

$$g(S)(\lambda - S) \subset (\lambda - S)g(S) = s - f(S).$$

Since $\lambda \in \sigma(S)$, $\lambda - S$ is either not 1-1 or not onto, so the same is true of $x \in \sigma(f(S))$. Thus $f(\tilde{\sigma}(S)) \subset \sigma(f(S))$.

Next, we show isometry and uniqueness. Recall that operator norm is specified its spectral. we get

$$||f(S)|| = \sup\{f(\lambda)\,;\ \lambda \in \sigma(\tilde{S})\} = ||f||,\ f \in \mathcal{F}$$

Therefore this map is an isometry of $\mathcal{F}$, considered as a subalgebra of $C(\tilde{\sigma}(S))$, into $\mathcal{B}(\mathcal{H})$. On $\sigma(S) \subset \mathbb{R}$, $r_\lambda^* = r_{\overline{\lambda}}$. Therefore $\mathcal{F}$ is a closed complex conjugation. The Stone-Weierstrass theorem show that $\mathcal{F}$ is dense in $C(\tilde{\sigma}(S))$, so $f \mapsto f(S)$has a unique extension to an algebra isometry.

Finally, we show the other relations. $(\overline{\lambda} - S^*)^{-1} = ((\lambda - S)^{-1})^*$ imply $r_\lambda^*(S) = r_{\overline{\lambda}} = r_\lambda(S)^*$. Thus $f^*(S) = f(S)^*$, $^\forall f \in \mathcal{F}$. This relation is carried to $C(\tilde{\sigma}(S))$. suppose $f \geq 0, f \in C(\tilde{\sigma}(S))$,, then there exists $g \in C(\tilde{\sigma}(S)), g \geq 0$, such that $f = g^2$. Then $f(S) = g(S)^2 = g(S)^* g(S) \geq 0$. Conversely if $f(S) \geq 0$, then $f(\tilde{\sigma}(S)) = \sigma(f(S)) \subset 0, \infty)$, (Recall that put a self adjoint operator S, then $\sigma(S) \subset [\alpha, \beta]$, where $\alpha = \inf\{(Sx, x)\,;\, ||x|| = 1\}, \beta = \sup\{(Sx, x)\,;\, ||x|| = 1\}$.) Thus $f \geq 0$. □

Corollary 2.3. S is a self adjoint operator. Then S is bounded if and only if $\sigma(S)$ is bounded.

Proof. If S bounded, $\sigma(S) \subset [\alpha, \beta]$, where $\alpha = \inf\{(Sx, x)\,;\, ||x|| = 1\}, \beta = \sup\{(Sx, x)\,;\, ||x|| = 1\}$ is bounded. Conversely suppose $\sigma(S)$ is bounded. Let $f(s) \equiv s$. Then $f \in C(\sigma(S))$. Moreover, $(f - \lambda) r_\lambda = e, \lambda \in \rho(S)$, so $(f(S) - \lambda) R_\lambda = R_\lambda (f(S) - \lambda) = I$. Therefore $f(S) - \lambda = S - \lambda$, so $S = f(S) \in \mathcal{B}(\mathcal{H})$.

□

Now, we have functional calculus which have only continuous functions, so we extends this calculus for some general functions.

Definition 2.4. First, let σ be a closed subset of $\mathbb{R}$, and let $\mathcal{A}_0(\sigma)$ be the algebra continuous complex valued function on σ which is 0 at ∞. Next, let $\mathcal{A}_+(\sigma)$ be the set of bounded functions on σ which are pointwise limits of increasing sequence of non-negative functions in $\mathcal{A}_0$. Finally, let $\mathcal{A}_1$ be the algebra generated by $\mathcal{A}_1$.

If σ is a bounded, $\mathcal{A}_0$ equal to algebra of continuous function $C(\sigma)$. All bounded continuous non-negative functions are written as pointwise limits of increasing sequence of non-negative functions, so $\mathcal{A}_+$ contains all of them. Similarly, $\mathcal{A}_1$ contained all bounded continuous function on σ.

Write $f_n \nearrow f$ is real valued and $\{f_n(s)\}_n$ is increase sequence, all $s \in \sigma$.

Definition 2.5. $\{S\} \subset \mathcal{B}(\mathcal{H})$ is called full if $\vee \mathcal{R}(S) = \mathcal{H}$

Note that if S is self adjoint, $\{(\lambda - S)^{-1}\}_\lambda$ is full. Then, next lemma is essential to extend functional calculus.

Lemma 2.6. Let σ is a closed subset in $\mathbb{R}$, and let $A_0 : \mathcal{A}_0(\sigma) \to \mathcal{B}(\mathcal{H})$ be an isometric algebra isomorphism which preserve adjoints. Then A_0 has a unique extension to an algebra homomorphism $A_1 : \mathcal{A}_1 \to \mathcal{B}(\mathcal{H})$ with property that if $\{f_n\} \subset \mathcal{A}_+$ and $f_n \nearrow f \in \mathcal{A}_+$, then $A_1(f_n) \to A_1(f)$. Moreover, the extention A_1 also preserving adjoint and order, and $||A_1(f)|| \leq \sup_{s \in \sigma} |f(s)|$. The image of A_0 is full if and only if $A_1(e) = I$.

Proof. First, the same argument in theorem 2.1 shows that A_0 preserves order.

Now, we will define $A_1(f)$ which has uniqueness step by step, and then proof the other relation. Suppose $0 \leq f_n \nearrow f$, where $f_n \in \mathcal{A}_0$, f is bounded. Then $\{A_0(f_n)\}$ is a bounded increase sequence of operators in $\mathcal{B}(\mathcal{H})$, so $A_0(f_n) \to S \in \mathcal{B}(\mathcal{H})$. Suppose also $g_n \in \mathcal{A}_0$, $0 \leq g_n \nearrow f$. Then $A_0(g_n) \to T \in \mathcal{B}(\mathcal{H})$. Let $f_n \wedge g_m(s) := \min\{f_n(s), g_m(s)\}$. Then $f_n \wedge g_m \nearrow f_n$ as $m \to \infty$. These functions are continuous and 0 at ∞ (if σ is unbounded), so the convergence is uniform. Thus $T = \lim A_0(g_m) \geq \lim A_0(f_n \wedge g_m) = A_0(f_n)$, all n, so $T \geq S$. Also we get $S \geq T$. Therefore the limit$S = T =: A_1(f)$ is independent of the approximating sequence. If $f \in \mathcal{A}_0 \cap \mathcal{A}_+$, set $f_n \equiv f$. Then $A_1(f) = \lim A_0(f_n) = A_0(f)$.

Suppose $f_n, g_n \in \mathcal{A}_0$, $0 \le f_n \nearrow f \in \mathcal{A}_+$, $0 \le g_n \nearrow g \in \mathcal{A}_+$. Then $f_n + g_n \nearrow f + g$, $f_n g_n \nearrow fg$, so $A_1(f+g) = A_1(f) + A_1(g)$, $A_1(fg) = A_1(f)A_1(g)$. Similarly, if $f \in \mathcal{A}_+$, $\alpha \ge 0$, then $A_1(\alpha f) = \alpha A_1(f)$. If $f \le g$, $f, g \in \mathcal{A}_+$, then we put the sequence $f_n \le g_n$. Thus $A_1(f) = \lim A_0(f_n) \le \lim A_0(g_n) = A_1(g)$.

Now, since $\mathcal{A}_+$ is closed under addition, multiplication, and multiplication by non-negative numbers, $\mathcal{A}_1$ contains precisely of all functions of the form $f = f_1 f_2 + i f_3 - i f_4$, $f_j \in \mathcal{A}_+$. Also if $f = g_1 - g_2 + i g_3 - i g_4, g_j \in \mathcal{A}_+$, then $f_1 + g_2 = g_1 + f_2, f_3 + g_4 = g_3 + f_4$. Thus by the algebraic property of A_1, $A_1(f_1) - A_1(f_2) + iA_1(f_3) - iA_1(f_4) = A_1(g_1) - A_1(g_2) + iA_1(g_3) - iA_1(g_4)$. Therefore A_1 has a unique linear extension to $\mathcal{A}_1$, and extension is an algebra homomorphism.Moreover, this argument also shows $A_1(f^*) = A_1(f)^*$. If $f, g \in \mathcal{A}_1$ and $f \ge g$, then $f = f_1 - f_2$, $g = g_1 - g_2$, $f_j, g_j \in \mathcal{A}_+$, and $f_1 + g_2 \ge g_1 + f_2$, so $A_1(f_1 + g_2) \ge A_1(g_1 + f_2)$. Therefore $A_1(f) \ge A_1(g)$.

$f_n \in \mathcal{A}_+$ and$f_n \nearrow f \in \mathcal{A}_+$, take sequences $\{f_{n,m}\} \subset \mathcal{A}_0 \cap \mathcal{A}_1$ with $f_{n,m} \nearrow f_n$. Let $g_n(s) := \max\{f_{1,n}(s), \cdots f_{n,n}(s)\}$. Then $g_n \in \mathcal{A}_0 \cap \mathcal{A}_+$ and$g_n \le f_n$, $g_n \nearrow f$. Therefore $A_1(g_n) \le A_1(f_n) \le A_1(f)$, and since $A_1(g_n) \to A_1(f)$ we have $A_1(f_n) \to A_1(f)$. If $|f(s)| \le c$, then $(f^* f)(s) \le c^2$, so $A_1(f)^* A_1(f) \le c^2 A_1(e) \le c^2 I$, so $||A_1(f)||^2 \le c^2$.

Finally, we discuss about fullness. if $\{A_0(f)\}$ is full, then since $A_1(e)A_1(f) = A_1(ef) = A_1(f)$, all f, we have $A_1(e) = I$. Conversely, let $\mathcal{H}_1 = \vee \mathcal{R}(A_0(f))$ be a subspace of $\mathcal{H}$. Then each $A_1(f)$ has range in $\mathcal{H}_1$. Therefore since $A_1(e) = I$, $\mathcal{H}_1 = \mathcal{H}$, this means fullness.

□

Finally, we extend functional calculus to the locally continuous function, so the image of functional calculus is extended to the unbounded densely defined closed operators.

Definition 2.7. Let e^N be the characteristic function of $(-N, N)$ and the restriction of e^n to σ (it is in $\mathcal{A}_+$) also denoted by e^N. Let $\mathcal{A}$ be the algebra of functions on σ which are locally in $\mathcal{A}_1$, i.e. those $f$such that $e^N f \in \mathcal{A}_1$, all $N > 0$. Write $f^N := e^N f$.

Theorem 2.8. Let σ be a closed subset of $\mathbb{R}$ and let $A_0 : \mathcal{A}_0(\sigma) \to \mathcal{B}(\mathcal{H})$ be an isometric algebra isomorphism onto a full subalgebra of $\mathcal{B}(\mathcal{H})$ of which preserve adjoints. There is an unique extension $A : \mathcal{A}(\sigma) \to \mathcal{C}(\mathcal{H})$, which has the properties:

1. $A(\alpha f) = \alpha A(f),\ \alpha \neq 0$;
2. $A(f) + A(g) \subset A(f+g)$;
3. $A(f)A(g) \subset A(fg)$;
4. $A(f^*) = A(f)^*$;
5. f is bounded$\Rightarrow ||A(f)|| \le \sup_\sigma |f(s)|$;
6. $f \ge 0 \Rightarrow (A(f)x, x) \ge 0,\ all\ x \in D(A(f))$;
7. $f_n \in \mathcal{A}_+,\ f_n \nearrow f \Rightarrow (A(f_n)x, x) \to (A(f)x, x)\ all\ x \in D(A(f))$;
8. if $s \in D(A(f))$, then $A(f^N)x \to A(f)x$ as $N \to \infty$.

Proof. Lemma 2.6 give us a unique extension $A_1 : \mathcal{A}_1 \to \mathcal{B}(\mathcal{H})$. Given $f \in \mathcal{A}$, let $D(A(f)) = \{x\,;\, \sup_N ||A_1(f^N)x|| < \infty\}$. Suppose $x \in D(A(f))$. $||A_1(f^N)x||^2 = (A_1(|f^N|^2)x, x)$ is bounded and non-decreasing with N, hence convergent as $N \to \infty$. For $M < N$, $(f^*)^M f^N = (f^*)^M f^M$, so

$$||A_1(f^N)x - A_1(f^M)x||^2 = ||A_1(f^N)x||^2 - ||A_1(f^M)||^2 \to 0 \ (M, N \to \infty).$$

Let $A(f)x$ be this limit. This is independent of choice of internal. By this extension, $A(f)$ is linear. Since $e^N \nearrow e$, $A_1(e^N)A_1(f) = A_1(f)$. Therefore if $f \in \mathcal{A}_1$, so that $D(A(f)) = \mathcal{H}$, we have

$$A(f) = \lim A_1(f^N) = \lim A_1(e^N)A_1(f) = A_1(f).$$

In general, if f is bounded then the f^N are uniformly bounded; by the lemma $||A_1(f^N)|| \leq \sup|f^N(s)| \leq \sup|f(s)|$. Therefore $A(f) \in \mathcal{B}(\mathcal{H})$, $||A(f)|| \leq \sup|f(s)|$.

Each $E^N := A(e^N)$ is projection whose range is clearly contained in $D(A(f))$ for any $f \in \mathcal{A}$. Since $E^N \to I$, $\vee E^N(\mathcal{H}) = \mathcal{H}$. Therefore each $A(f)$ is densely defined. Suppose $y \in D(A(f^*))$, $x \in D(A(f))$. Then

$$\begin{aligned} (y, A(f)x) &= \lim(y, A(f^N)x) \\ &= \lim(A(f^*)^N x, y) \\ &= (A(f^*)y, x). \end{aligned}$$

Conversely, suppose $y \in D(A(f)^*)$ and $A(f)^*y = z$. Then

$$\begin{aligned} (E^N y, A(f^N)x) &= (y, A(f)E^N x) \\ &= (z, E^N x) \\ &= (E^N z, x). \end{aligned}$$

Thus $A((f)^{*N})y = E^N z \to z$, so $y \in D(A(f^*)^*)$ and $A(f^*)y = z$. This proves 4 and it follows that $A(f) = A(f^*)^*$ is closed.

So far we have defined A, shown that $A(f) \in \mathcal{C}(\mathcal{H})$ and proved 4 and 7. Before show the other relations, let us show uniquness. If B is another extension of A_0, it also extends A_1. By 7, $B(f) \subset A(f)$. Taking adjoint, $B(f) = B(f^*)^* \supset A(f^*)^* = A(f)$. Therefore $A(f) = B(f)$.

Now, we show the other relations. 1 is trivial, Suppose $f, g \in \mathcal{A}$. $(f+g)^N = f^N + g^N$, $(fg)^N = f^N g^N$ and $||A_1((f+g)^N x)|| = ||A_1(f)x + A_1(g)x||$. Thus $D(A(f)) \cap D(A(g)) \subset D(A(f+g))$. For $x \in D(A(f)) \cap D(A(g))$,

$$A(f+g)x = \lim(A_1((f+g)^N x) = A(f)x + A(g)x.$$

If $x \in D(A(g))$ and $A(g)x \in D(A(f))$, then $||A((fg)^N)x|| = ||A(f^N)A(g)x||$ is bounded. Thus $D(A(f)A(g)) \subset D(A(fg))$ and since $A_1(f^N)A_1(g^M) = A_1((fg)^N)$, $N < M$,

$$\begin{aligned} A(f)A(g)x &= \lim A_1(f^N)A(g)x \\ &= \lim A_1(f^N)(\lim A_1(g^N)x) \\ &= \lim A_1((fg)^N x) \\ &= A(fg)x. \end{aligned}$$

If $f \geq 0$, then $f^N \geq 0$, so 6 follows fram that A_1 preserves order. Finally, suppose $f_n \in \mathcal{A}_+, f_n \nearrow f \in \mathcal{A}$. Then for $x \in D(A(f))$,

$$(A(f_n)^N x, x) \leq (A(f_n)x, x) \leq (A(f)x.x).$$

By the Lemma 2.6, $A(f_n^N) \to A(f^N)$, and since $(A(f^N)x, x) \to (A(f)x, x)$, we get 7.

□

Corollary 2.9. *S is a self adjoint operator and σ is$\sigma(S)$. Let $A_0 : \mathcal{A}_0(\sigma) \to \mathcal{B}(\mathcal{H})$ be given by $A_0(f) = f(S)$. Let A be the extension of theorem, and $f(s) \equiv s$. Then $A(f) = S$.*

Proof. Let $h(s) = (s - i)^{-1}$. Then $hf = 1 + if$, so $(S - i)^{-1}A(f) \subset I + i(S - i)^{-1}$. Multiplication on the left by $S - i$ gives $A(f) \subset S$. Then $A(f) = A(f)^* \supset S^* = S$, so $A(f) = S$.

□

Conbining Theorem 2.2, 2.8, and Corollary 2.9, we have a map from $\mathcal{A}(\sigma(S))$ to $\mathcal{C}(\mathcal{H})$, which we shall denoted by $f \mapsto f(S)$. This is an abstract construction of Functional Calculus.

Next, we see explicit construction of this functional calculus.

Proposition 2.10. *Let $\{E_t\}$ be a spectral measure with support σ and define $e_t(s) := 1,\ s \leq t, e_t(s) := 0,\ s > t$. The map $e_t \mapsto E_t$ has extension, written $f \mapsto \int f(s)dE_s\ : \mathcal{A}(\sigma) \to \mathcal{C}(\mathcal{H})$ and it satisfy the same consitions of theorem 2.8.*

Proof. If B is a half-open interval $(a, b]$, let e^B be the characteristic function. Then $e^B = e_b - e_a$. Let $E(B) = E_b - E_a$, and let $E(\emptyset) = 0$. If B_1, B_2 are two such intervals, then the fact that E_t increases with t implies $E(B_1 \cap B_2) = E(B_1)E(B_2)$, while $E(B_1 \cup B_2) = E(B_1) + E(B_2) - E(B_1 \cap B_2)$, if $B_1 \cup B_2$ is again an interval. It follows that the map $e^B \to E(B)$ has a unique linear extension which is an algebra homomorphism from the algebra of functoins generated by the e^B into $\mathcal{B}(\mathcal{H})$. Furthermore if $f = \sum \alpha_j e^{B_j}$ where the B_j are disjoint and the sum is finitem then for $x \in \mathcal{H}$,

$$\begin{aligned} ||\sum \alpha_j E(B_j)x||^2 &= \sum |\alpha_j|^2 ||E(B_j)x||^2 \\ &\leq \sup\{|\alpha_j|\ ;\ R(B_j) \neq 0\} \sum ||E(B_j)x||^2 \\ &\leq \sup_\sigma\{|f(s)|\}||x||^2. \end{aligned}$$

Conversely, $\sup_\sigma\{|f(s)|\} = |\alpha_k|$ for some k such that $E(B_k) \neq 0$. Take x in the range of $E(B_k)$with $||x|| = 1$. Then $||\sum \alpha_j E(B_j)x||^2 = ||\alpha_k x||^2 = |\alpha_k|^2$. Thus the map $f \to \sum \alpha_j E(B_j)$ has a unique extension to an algebra isometry into $\mathcal{B}(\mathcal{H})$ from the algebra $\mathcal{A}_*(\sigma)$which is the closure in the sup norm on σ of the algebra generated by the e_B. Moreover, this map is seen to preserve adjoint.

Since $\mathcal{A}_0(\sigma) \subset \mathcal{A}_*(\sigma)$ we may restrict to $\mathcal{A}_0(\sigma)$ and apply Theorem 2.2 to get a mapping $A_1 : \mathcal{A}_1(\sigma) \to \mathcal{B}(\mathcal{H})$. We assert that $A_1(e_t) = E_t$. Given $t \in \mathbb{R}$, choose a sequence $\{f_n\}$ of continuous functions with $0 \leq f_n \leq e$, $f_n \nearrow$,$f_n(s) = 0,\ s \leq t,\ f_n(s) = 1,\ t + \dfrac{1}{n} \leq s \leq t + n$, and $f_n(s) = 0,\ s \geq t + n + 1$. Then $f_n \nearrow 1 - e_t$. Let $B_n = (t + \dfrac{1}{n}, t + n]$, and let $F_n := A_1(f_n)$. Our construction gives $E(B_n) \leq F_n \leq I - E_t$. However, $E(B_n) \to I - E_t$, so $A_1(1 - e_t) = \lim A_1(f_n) = I - E_t$, or $A_1(e_t) = E_t$. Thus $A/1$ extends the map $e_t \to E_t$; moreover $E_t \nearrow I$ as $t \to \infty$ implies that $A_1(\mathcal{A}_0)$ is full. The assertion now follows from Theorem 2.2.

□

Proposition 2.11. *Let $\{E_t\}$ be a spectral measure and let*

$$U_t := \int e^{its} dE_t.$$

Then $\{U_t\}$ is strongly continuous unitary group.

Proof. Let $u_t(s) := e^{ist}$. $U_t = \int u_t \, dE_s$ is in the image of extension map $f \mapsto \int f(s) \, dE_s$ in Proposition 2.10. Therefore U_t satisfies corresponding properties of u_t. $u_t^* u_t = 1$ shows each U_t is unitary, and $u_{s+t} = u_t u_s$ shows $U_{s+t} = U_s U_t$. Now we have already $\vee \mathcal{R}(E_b - E_a) = \mathcal{H}$. Put $x \in \mathcal{R}(E_b - E_a)$, then $u_t(e_b - e_a) \to u_s(e_b - e_a)$ uniformly as $t \to s$. Thus $U_t x \to U_s x$ everywhere. □

Proposition 2.12. Let $\{U_t\}$ be a strongly continuous unitary group in $\mathcal{H}$ and let $S := \lim_{t\to 0} \dfrac{1}{it}(U_t x - x)$. Then S is exists and self adjoint.

Proof. The existence is following meaning. $\{U_t\}$ is unitary group, so the domain $D(S)$ is defined as the set of all x such that $\lim_{t\to 0}(U_t x - x)$ is exists, and let$Sx = y$. This is well-defined.

Now, we proof self-adjointness. Suppose ϕ is a continuous function on $[0, \infty)$ with $\int_0^\infty |\phi(s)| ds < \infty$. Define $T_\phi x,\ x \in \mathcal{H}$, as

$$T_\phi x = \int_0^\infty \phi(s) U_s x ds.$$

Since $U_t x$is continuous about t and has norm $||x||$, this show the existence of the improper Riemann integral, and so $T_\phi \in \mathcal{B}(\mathcal{H})$. Suppose also that the derivation $\phi'$is continuous and absolutely integrable on $[0, \infty)$. For $t > 0$,

$$\begin{aligned}\frac{1}{it}(U_t - I)T_\phi x &= \frac{1}{it}\int_0^\infty \phi(s)[U_{s+t}x - U_s x]ds \\ &= \int_t^\infty \frac{1}{it}[\phi(s-t) - \phi(s)]U_s x ds + \frac{1}{it}\int_0^t \phi(s) U_s x ds.\end{aligned}$$

Then we think about limit this fomula, and since $U_s x \to x$ as $s \to 0$ from Proposition 2.11,

$$\begin{aligned}\lim_{t\to 0}\frac{1}{it}(U_t - I)T_\phi x &= \lim_{t\to 0}\int_t^\infty \frac{1}{it}[\phi(s-t) - \phi(s)]U_s x ds + \frac{1}{it}\int_0^t \phi(s) U_s x ds \\ &= iT_{\phi'} + i\phi(0)I,\end{aligned}$$

so we get $ST_\phi = iT_{\phi'} + i\phi(0)I$. If $\{\phi_n\}$ is a sequence of functions as above with $\int_0^\infty \phi_n(s) ds = 1, \phi_n(s) = 0$ for $s \geq \dfrac{1}{n}$, then $T_{\phi_n} x \to$ as $n \to \infty$. Thus $D(S)$ is dense.

Since $(\dfrac{1}{it}(U_t - I)x, y) = (y, \dfrac{1}{it}(U_t - I)x)$, S is symmetric, so we should show only that $\pm i \in \rho(S)$ from proposition 2.1. Take $\phi(s) = ie^{-s}$ and consider $ST_\phi = iT_{\phi'} + i\phi(0)I$. Define R_+ as

$$R_+ x = -i\int_0^\infty e^{-s} U_s x ds.$$

Then

$$\begin{aligned}(S+i)R_+ x &= (S+i) - i\int_0^\infty e^{-s} U_s x ds \\ &= -iST_\phi x + \int_0^\infty e^{-s} U_s x ds \\ &= iT_{\phi'} x + i\phi(0)Ix - iT_{\phi'} s \\ &= Ix,\end{aligned}$$

so $(S+i)R_+ = I$ and the domain is everywhere. Since $U_tU_s = U_sU_t$ from Proposition2.12, $D(S)$ is invariant under U_t and $U_tS \subset SU_t$. It follows that $R_+(S+i) \subset (S+i)R_+ = I$ Therefore $-i \in \rho(S)$ and $R_+ = (S+i)^{-1}$. Similarly, $i \in \rho(s)$ and $(S-i)^{-1} = R_-$, where

$$R_-x = i\int_0^\infty e^{-s}U_{-s}sds.$$

Thus S is self adjoint.

□

Theorem 2.13. There are 1-1 correspondence between self adjoint operators S, spectral measure $\{E_t\}$ and unitary groups $\{U_t\}$ given by:

$$E_t = e_t(S),\ S = \int s\, dE_s;$$

$$U_t = u_t(S),\ S = \lim_{s\to 0}\frac{1}{is}(U_s - I);$$

$$U_t = \int e^{ist}dE_s.$$

Proof. We show first correspondence. If S is self adjoint and $E_t = e_t(S)$, then by uniqueness of the extensions to $\mathcal{A}(\mathbb{R})$ and the corollary of theorem 2.8, $S = f(s) = \int f(s)dE_s$, where$f(s) = s$. Conversely, if $\{E_t\}$ is a spectral measure and $S = \int s dE_s$, them $(\lambda - S)^{-1} = \int r_\lambda(s)dE_s$, $\lambda \notin \mathbb{R}$; this follows from the properties of the map $g \mapsto \int g(s)dE_s$. Since the $\{r_\lambda\}$ generate a dense subalgebra of $\mathcal{A}_0(\sigma)$, we get $g(S) = \int g(S)dE_s$ for every $g \in \mathcal{A}_0(\sigma)$ and therefore for every $g \in \mathcal{A}_1(\sigma)$. In particular, $E_t = \int e_t(S)dE_s = e_t(S)$.

Now, we show second correspondence. If $U_t = u_t(S)$ and $T = \lim \frac{1}{it}(U_t - I)$, let $\{E_t\}$ be the spectral measure associated with S. Now $\frac{1}{i\epsilon}(u_\epsilon - e)(e_b - e_a) \to f(e_b - e_a)$ unifomly as $\epsilon \to 0$, where $f(s) = s$. Therefore the range of $E_b - E_a$, $a < b$, is in $D(T)$ and $T = S$ on this range. For any $x \in D(S) \to 0$, $x_N := (E_N - E_{-N})x \to x$ and $Tx_N = Sx_N \to Sx$ as $N \to \infty$. Therefore $S \subset T$. However, $S = S^* \subset T^* = T$, so $S = T$. Conversely, let $\{U_t\}$ be a unitary group, $S = \lim \frac{1}{it}(U_t - I)$, and $V_t = u_t(S)$. Then we have also $S = \lim \frac{1}{it}(V_t - I)$. If $x, y \in D(S)$, then $U_tx, V_ty \in D(S)$, all t. Therefore the scalar function$\phi(t) = (U_tx, V_ty)$ is differentiable, and

$$\phi'(t) = (iSU_tx, V_xy) + (U_tx, iSV_ty) = 0.$$

Thus $(U_tx, V_ty) \equiv (U_0x, V_0y) \equiv (x, y)$. Since $D(S)$ is dence, this implies $V_t^*U_t = I$, or $V_t = U_t$, i.e. $\{U_t\}$ and $\{V_t\}$ are equivalent.

Finally, we show third correspondence. If$U_t = \int e^{its}dE_s$ where $\{E_t\}$ is a spectral measure. Let $S = \int s\, dE_s$. Then we know $U_t = u_t(S)$, and consequently $S = \lim \frac{1}{it}(U_t - I)$. Then we know $U_t = u_t(S)$, and consequently $S = \lim \frac{1}{it}(U_t - I)$. Therefore we have already $\{U_t\}$ and $\{E_t\}$ uniquely from S.

□

This is the explicit construction of functional calculus. For example, we want to calculous $f(S)$, we calculus $\int f(s)dE_s$. Moreover, this construction reveal the representation of S as unitary group. For example, we have $S_0 f := if'$ on C^1, and S is closure of S_0. Then S is self adjoint unbounded operator. From theorem 2.12, there exists unitary group$\{U_t\}$ which correspond to S. $\{U_t\}$ must satisfy the relation $U_t = u_t(S)$, so we get $U_t f(s) = f(s-t)$ by simple calculus of Taylor's formula. Therefore U_t is a shift operator.

3 Reference

[1] The lecture notes of this class
[2] R. Beals, "topics in operator theory", The University of Chicago Press Chicago and London, (1971).