Homework 3

Zhang Liyang

1. (Lecture 2) Let G be a finite group of transformation of X, then
$\forall x\in X: |S(x)|\cdot|O(x)| = |G|$

Pf: If $|O(x)|=\infty$, then $|G|=\infty$. $\Rightarrow |O(x)|<\infty$. And since $S(x)\subset G$, $|S(x)|<\infty$.
Denote $n:=|O(x)|$ and $O(x)=\{x_1,x_2,\cdots,x_n\}$, and
$G_j:=\{g\in G \mid g(x_1)=x_j\}$ $\forall j\in\{1,2,\cdots,n\}$. Hence $G=G_1\sqcup G_2\sqcup\cdots\sqcup G_n$.

$\Rightarrow G_1=S(x_1)$, $|G|=|G_1|+|G_2|+\cdots+|G_n|$

$\forall j\in\{2,3,\cdots,n\}\ \forall g_j\in G_j: S(x_j)g_j=\{sg_j\in G \mid s(x_j)=x_j\}=\{sg_j\in G \mid sg_j(x_1)=x_j\}=G_j$

$\Rightarrow |S(x_j)| = |S(x_j)g_j| = |G_j|$

$\Rightarrow |G|=|G_1|+|G_2|+\cdots+|G_n| = |S(x_1)|+|S(x_2)|+\cdots+|S(x_n)|$

$\forall j,k\in\{1,2,\cdots,n\}\ \exists a\in G\ \forall s_j\in S(x_j): a(x_j)=x_k \Leftrightarrow a^{-1}(x_k)=x_j$

$\Rightarrow as_ja^{-1}(x_k)=as_j(x_j)=x_k \Rightarrow as_ja^{-1}\in S(x_k)$

$\Rightarrow \exists\phi: S(x_j)\ni s_j\longmapsto as_ja^{-1}\in S(x_k)$

$\Rightarrow \exists\phi^{-1}: S(x_k)\ni s_k\longmapsto bs_kb^{-1}\in S(x_j)$ since j,k are arbitrary

$\Rightarrow \phi$ is invertible

$\forall s_{j1},s_{j2}\in S(x_j): \phi(s_{j1}s_{j2})=as_{j1}s_{j2}a^{-1}=as_{j1}a^{-1}as_{j2}a^{-1}=\phi(s_{j1})\phi(s_{j2})$

$\Rightarrow \phi$ is isomorphism between $S(x_j)\leftrightarrow S(x_k)$

$\Rightarrow S(x_j)\Leftrightarrow S(x_k) \Rightarrow |S(x_j)|=|S(x_k)|$

$\Rightarrow |G|=\sum_{j=1}^{n}|S(x_j)| = n|S(x_1)| = |S(x_1)|\cdot|O(x_1)|$

$\Rightarrow \forall x\in X: |G|=|S(x)|\cdot|O(x)|$ since x_1 is arbitrary $\square$

2. Let G be a subgroup of $O(3)$ but not a subgroup of $SO(3)$. Then $\underset{O(3)}{G} = \underset{SO(3)}{G_+} \sqcup \underset{O(3)\setminus SO(3)}{G_-}$.
In the case $-\mathbb{1}=:\Pi\notin G$, prove $|G_+|=|G_-|$.
(it is appearant when $\Pi\in G$) det $= \pm1 \quad +1 \quad -1$

Pf: $G\not\subset SO(3) \Rightarrow \exists p\in G_- \Rightarrow \det(p)=-1$

$\forall a\in G_+: \det(a)=1$, $\det(pa)=\det(p)\det(a)=-1 \Rightarrow pa\in G_-$

$\Rightarrow pG_+\subset G_- \Rightarrow |G_+|=|pG_+|\leq|G_-|$ ①

$\forall b\in G_-: \det(b)=-1$, $\det(pb)=\det(p)\det(b)=1 \Rightarrow pb\in G_+$

$\Rightarrow pG_-\subset G_+ \Rightarrow |G_-|=|pG_-|\leq|G_+|$ ②

①② $\Rightarrow |G_+|=|G_-|$ $\square$

3. Find all finite subgroups of $SO(2)$ and $O(2)$ which leave a lattice invariant.
Consider G finite non-trivial subgroup of $SO(2)$, and set
$S^1:=\{n\in\mathbb{R}^2 \mid \|n\|=1\}$ (since I cannot use $\mathbb{C}$ for circle)

$\forall R\in O(2)$, set $R=\begin{pmatrix} a_1 & b_1\\ a_2 & b_2\end{pmatrix}$, then $\begin{cases} a_1b_1+a_2b_2=0 & ①\\ a_1^2+a_2^2=b_1^2+b_2^2=1 & ②\\ a_1b_2-b_1a_2=\pm1 & ③\end{cases}$ ① $\Rightarrow \frac{b_2}{a_1}=-\frac{b_1}{a_2}=:k$ (if $a_1\neq0$, $a_2\neq0$)

$\Rightarrow R=\begin{pmatrix} a_1 & -ka_2\\ a_2 & ka_1\end{pmatrix} \overset{③}{\Rightarrow} ka_1^2+ka_2^2=\pm1 \overset{②}{\Rightarrow} k=\pm1=\begin{cases}+1 & \text{if } A\in SO(2)\\ -1 & \text{if } A\in O(2)\setminus SO(2)\end{cases}$

If $a_1=0$, $R=\begin{pmatrix} 0 & \pm1\\ \pm1 & 0\end{pmatrix}$; If $a_2=0$, $R=\begin{pmatrix}\pm1 & 0\\ 0 & \pm1\end{pmatrix}$.

$\Rightarrow$ In all cases R can be expressed as $\begin{pmatrix} a_1 & -ka_2\\ a_2 & ka_1\end{pmatrix}$ with $k=\pm1$.

Known $R=\begin{pmatrix} a_1 & -ka_2 \\ a_2 & ka_1 \end{pmatrix}$ and $a_1^2+a_2^2=1$, $\exists\theta\in(-\pi,\pi]$: $R=\begin{pmatrix} \cos\theta & -k\sin\theta \\ \sin\theta & k\cos\theta \end{pmatrix}$ with $k=\pm1$.

$\Rightarrow tr(R)=2\cos\theta$ when $R\in SO(2)$ and $tr(R)=0$ when $R=O(2)$

Recall $\mathcal{L}=\{c_1b_1+c_2b_2 \mid c_j\in\mathbb{Z}\}$ with $\underline{b}:=(b_1,b_2)$ basis of $\mathbb{R}^2$

$\Rightarrow [R]_{\underline{b}}=\begin{pmatrix} c_{11} & c_{21} \\ c_{12} & c_{22} \end{pmatrix}$ with $c_{11},c_{12},c_{21},c_{22}\in\mathbb{Z}$ $\therefore tr(R)\in\mathbb{Z}$

(1) $\Rightarrow$ In the case $R\in SO(2)$, $2\cos\theta\in\mathbb{Z}$ $\Rightarrow$ $\theta\in\{0,\pm\frac{\pi}{3},\pm\frac{\pi}{2},\pm\frac{2\pi}{3},\pi\}$
which means there are only 8 elements of $SO(2)$
that leave a lattice invariant.

Denote them by $R_\theta:=\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ with $\theta\in\{0,\pm\frac{\pi}{3},\pm\frac{\pi}{2},\pm\frac{2\pi}{3},\pi\}$ (✻)

$\Rightarrow$ All finite subgroups of $SO(2)$ leaving a lattice invariant are:

$C_1:=\{R_0\}$

$C_2:=\{R_0,R_\pi\}$

$C_3:=\{R_0,R_{2\pi/3},R_{-2\pi/3}\}$

$C_4:=\{R_0,R_{\pi/2},R_\pi,R_{-\pi/2}\}$

$C_6:=\{R_0,R_{\pi/3},R_{2\pi/3},R_\pi,R_{-2\pi/3},R_{-\pi/3}\}$

(2) Next for $O(2)$.

Denote $\Pi=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, then $\begin{cases} \Pi\in O(2) \\ \Pi\notin SO(2) \end{cases}$ and $O(2)=SO(2)\otimes\{\mathbb{1},\Pi\}$ (✻)

Set $G=G_+\sqcup G_-$ s.t. $G_+=G\cap SO(2)$ and $G_-=G\setminus G_+$. Either $G\ni\Pi$ or $G\not\ni\Pi$.

(2a) When $G\ni\Pi$, there are 5 new cpg made of $G_+\sqcup\Pi G_+$
with $G_+\in\{C_1,C_2,C_3,C_4,C_6\}$.

(2b) When $G\not\ni\Pi$, set $\phi: G\mapsto SO(2)$, $\phi(R)=\begin{cases} R & \text{if } R\in G_+ \\ \Pi R & \text{if } R\in G_- \end{cases}$ and $\mathbb{G}=\phi(G)$.

$\Rightarrow G_+\cap\Pi G_-=$ empty set and ϕ is isomorphism between $G\leftrightarrow\mathbb{G}$

By the same argument of Question 2, $|G_+|=|G_-|$

$\Rightarrow \mathbb{G}$ is a finite subgroup of $SO(2)$

By inspection, the possible pairs of $(G_+,\mathbb{G})$ are
(C_1,C_2) (C_2,C_4) (C_3,C_6)

Conclusion: 13 finite subgroups

$C_1=\{R_0\}$; $C_2=\{R_0,R_\pi\}$; $C_3=\{R_0,R_{\frac{2\pi}{3}},R_{-\frac{2\pi}{3}}\}$; $C_4=\{R_0,R_{\frac{\pi}{2}},R_\pi,R_{-\frac{\pi}{2}}\}$;

$C_1\sqcup\Pi C_1$; $C_2\sqcup\Pi C_2$; $C_3\sqcup\Pi C_3$; $C_4\sqcup\Pi C_4$;

$C_2'=\{R_0,\Pi R_\pi\}$; $C_4'=\{R_0,\Pi R_{\frac{\pi}{2}},R_\pi,\Pi R_{-\frac{\pi}{2}}\}$;

$C_6=\{R_0,R_{\frac{\pi}{3}},R_{\frac{2\pi}{3}},R_\pi,R_{-\frac{2\pi}{3}},R_{-\frac{\pi}{3}}\}$

$C_6\sqcup\Pi C_6$

$C_6'=\{R_0,\Pi R_{\frac{\pi}{3}},R_{\frac{2\pi}{3}},\Pi R_\pi,R_{-\frac{2\pi}{3}},\Pi R_{-\frac{\pi}{3}}\}$

(Sorry I tried a whole night to find the 230 subgroups of $E(3)$ but I failed when it became hard to think at T and O)