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Groups and their Representations - Report

THE POINCARÉ GROUP AND SOME APPLICATIONS IN QUANTUM FIELD THEORY

Summary: We introduce the Poincaré group starting from Lorentz transformations of the coordinates and derive the Lie algebra of its proper orthochronous subgroup. We obtain the general (A,B) irreps of the homogeneous Lorentz group and describe their associated fields in Q.F.T

I - The Poincaré Group and Algebra

1) Lorentz Transformations

A Lorentz transformation is a coordinate transformation $x^\mu \mapsto x'^\mu$ that is linear:

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$$

(Using Einstein's summation convention for repeated indices)

with $a \in \mathbb{R}^4$ and $\Lambda \in M_4(\mathbb{R})$ leaving the metric $\eta = diag(\underset{\eta_{00}}{-1}, \underset{\eta_{11}}{1}, \underset{\eta_{22}}{1}, \underset{\eta_{33}}{1})$ invariant:

$$\eta_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = \eta_{\rho\sigma} .$$

↳ because the scalar product of 2 4-vectors A and B is $A.B = \eta_{\mu\nu} A^\mu B^\nu := A^\mu B_\mu$

Performing a second Lorentz Transf. (L.T.): $x'^\mu \mapsto x''^\mu = \bar{\Lambda}^\mu{}_\rho x'^\rho + \bar{a}^\mu$, we obtain:

$$x''^\mu = \underbrace{\left(\bar{\Lambda}^\mu{}_\rho \Lambda^\rho{}_\nu\right)}_{:= \tilde{\Lambda}^\mu{}_\nu} x^\nu + \underbrace{\left(\bar{\Lambda}^\mu{}_\rho a^\rho + \bar{a}^\mu\right)}_{:= \tilde{a}^\mu} .$$

Noting that

$$\begin{aligned} \eta_{\mu\nu} \tilde{\Lambda}^\mu{}_\rho \tilde{\Lambda}^\nu{}_\sigma &= \eta_{\mu\nu} \left(\bar{\Lambda}^\mu{}_\alpha \Lambda^\alpha{}_\rho\right)\left(\bar{\Lambda}^\nu{}_\beta \Lambda^\beta{}_\sigma\right) \\ &= \eta_{\alpha\beta} \Lambda^\alpha{}_\rho \Lambda^\beta{}_\sigma \\ &= \eta_{\rho\sigma} , \end{aligned}$$

we see that $(\tilde{\Lambda}, \tilde{a})$ defines a new L.T. from x^μ to x''^μ.

Furthermore, $(\mathbb{1}_4, 0_{\mathbb{R}^4})$ is a possible L.T., and clearly is an identity element for the multiplication law defined as:

$$(\Lambda, a) \cdot (\bar{\Lambda}, \bar{a}) = (\Lambda\bar{\Lambda}, \Lambda\bar{a} + a),$$

while the condition $\Lambda^t \eta \Lambda = \eta$ implies that $(\mathrm{Det}\,\Lambda)^2 = 1 \neq 0$, so that Λ is invertible. Then, the inverse of (Λ, a) is given by

$$(\Lambda, a)^{-1} = (\Lambda^{-1}, -\Lambda^{-1} a)$$

2) The proper orthochronous Lorentz group

The above information justifies the definition of the Poincaré group P given in lecture, where $P := \{(\Lambda, b) \mid \Lambda \in M_4(\mathbb{R})$ s.t. $\Lambda^t \eta \Lambda = \eta$ and $b \in \mathbb{R}^4\}$. The elements of P with $b = 0$ form an obvious subgroup called the (homogeneous) Lorentz group, denoted by "L" in lecture. We can also write $L = SO(3,1)$ ($\neq SO(4)$!)

The L.T. with $\mathrm{Det}(\Lambda) = +1$ also form a subgroup of P (or L), since $\mathrm{Det}(\Lambda\Lambda') = \mathrm{Det}(\Lambda)\,\mathrm{Det}(\Lambda')$. From $\eta_{\mu\nu} \Lambda^\mu{}_0 \Lambda^\nu{}_0 = \eta_{00} = -1$, we obtain:

$$(\Lambda^0{}_0)^2 = 1 + \sum_{i=1}^{3} (\Lambda^i{}_0)^2 = 1 + \sum_{i=1}^{3} (\Lambda^0{}_i)^2$$

because we also have $\Lambda^\nu{}_\sigma \Lambda^\kappa{}_\tau \eta^{\sigma\tau} = \eta^{\nu\kappa}$, where $(\eta^{\nu\kappa})^{-1} := \eta_{\nu\kappa}$

It implies that either $\Lambda^0{}_0 \geq +1$ or $\Lambda^0{}_0 \leq -1$. Consider Λ and $\bar{\Lambda}$ with $\Lambda^0{}_0 \geq 1$. We have

$$(\bar{\Lambda}\Lambda)^0{}_0 = \bar{\Lambda}^0{}_0 \Lambda^0{}_0 + \bar{\Lambda}^0{}_i \Lambda^i{}_0 = \bar{\Lambda}^0{}_0 \Lambda^0{}_0 + \vec{\bar{\Lambda}^0} \cdot \vec{\Lambda}_0,$$

usual inner product in $\mathbb{R}^3$

where $\vec{\bar{\Lambda}^0} = (\bar{\Lambda}^0{}_1, \bar{\Lambda}^0{}_2, \bar{\Lambda}^0{}_3)$ with $|\vec{\bar{\Lambda}^0}| = \sqrt{\sum_{i=1}^{3} (\bar{\Lambda}^0{}_i)^2} = \sqrt{(\bar{\Lambda}^0{}_0)^2 - 1}$ and similarly for $\vec{\Lambda}_0$. By the Cauchy-Schwarz inequality,

$$|\bar{\Lambda}^0{}_i \Lambda^i{}_0| = |\vec{\bar{\Lambda}^0} \cdot \vec{\Lambda}_0| \leq \sqrt{(\bar{\Lambda}^0{}_0)^2 - 1}\ \sqrt{(\Lambda^0{}_0)^2 - 1},$$

so that $(\bar\Lambda\Lambda)^0{}_0 \geq \bar\Lambda^0{}_0 \Lambda^0{}_0 - |\bar\Lambda^0{}_i \Lambda^i{}_0|$ implies:

$$(\bar\Lambda\Lambda)^0{}_0 \geq \bar\Lambda^0{}_0 \Lambda^0{}_0 - \sqrt{(\bar\Lambda^0{}_0)^2 - 1}\,\sqrt{(\Lambda^0{}_0)^2 - 1}.$$

Since $\bar\Lambda^0{}_0 \geq 1$ and $\Lambda^0{}_0 \geq 1$, $\exists\, x, y \in \mathbb{R}_{\geq 0}$ s.t. $\bar\Lambda^0{}_0 = \cosh(x)$ and $\Lambda^0{}_0 = \cosh(y)$.
Then,

$$(\bar\Lambda\Lambda)^0{}_0 \geq \cosh(x)\cosh(y) - \overbrace{|\sinh(x)|}^{=\sinh(x)}\overbrace{|\sinh(y)|}^{=\sinh(y)} = \cosh(x-y) \geq 1,$$

which shows that the subset of P (or L) with $\mathrm{Det}\,\Lambda = +1$ and $\Lambda^0{}_0 \geq 1$ is a subgroup. It is called the <u>inhomogeneous proper orthochronous Lorentz group</u>.
The subgroup of L would be "homogeneous"; it is denoted by $SO^+(3,1)$.

3) The Poincaré Algebra

The elements of the proper orthochronous Lorentz group are continuously connected to the identity (unlike those of the subsets of P with $\mathrm{Det}(\Lambda) = -1$ and/or $\Lambda^0{}_0 \leq -1$), so we can derive the Lie algebra by considering a proper orthochronous L.T. near the identity i.e.

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu, \quad a^\mu = \varepsilon^\mu,$$

where the $\omega^\mu{}_\nu$ and $\varepsilon^\mu{}_\nu$ are all infinitesimal. The condition $\Lambda^t \eta \Lambda = \eta$ then gives:

$$\begin{aligned}\eta_{\rho\sigma} &= \eta_{\mu\nu}(\delta^\mu{}_\rho + \omega^\mu{}_\rho)(\delta^\nu{}_\sigma + \omega^\nu{}_\sigma) \\ &= \eta_{\sigma\rho} + \omega_{\sigma\rho} + \omega_{\rho\sigma} + O(\omega^2),\end{aligned}$$

where $\eta_{\mu\sigma}\omega^\mu{}_\rho := \omega_{\sigma\rho}$. $\omega_{\mu\nu}$ is therefore antisymmetric (to first order).

As we saw in class, the subgroup of P connected to the identity can be unitarily represented by some representation U:

$$U(\mathbb{1} + \omega, \varepsilon) = \mathbb{1} + \frac{1}{2} i\omega_{\rho\sigma} J^{\rho\sigma} - i\varepsilon_\rho P^\rho + \dots,$$

where the coefficient operators $J^{\rho\sigma}$ and P^ρ must be Hermitian (i.e. self-adjoint) for U to satisfy the unitarity condition $U^{\dagger}U = UU^{\dagger} = \mathbb{1}$ (adjoint) to first order in ω and ε. Furthermore, $J^{\rho\sigma}$ should be antisymmetric because $\omega_{\rho\sigma} J^{\sigma\rho} \underset{\text{rename}}{=} \omega_{\sigma\rho} J^{\rho\sigma} = -\omega_{\rho\sigma} J^{\rho\sigma}$.

Let us first find the transformation properties of $J^{\rho\sigma}$ and P^{ρ}. We have:

$$
\begin{aligned}
U(\Lambda,a)U(\mathbb{1}+\omega,\varepsilon)U^{-1}(\Lambda,a) &= U(\Lambda(\mathbb{1}+\omega), \Lambda\varepsilon + a)\, U(\Lambda^{-1}, -\Lambda^{-1}a) \\
&= U(\Lambda(\mathbb{1}+\omega)\Lambda^{-1}, \Lambda(\mathbb{1}+\omega)(-\Lambda^{-1}a) + \Lambda\varepsilon + a) \\
&= U(\mathbb{1}+\Lambda\omega\Lambda^{-1}, \Lambda\varepsilon - \Lambda\omega\Lambda^{-1}a)
\end{aligned}
$$

To first order in ω and ε, we get:

$$U(\Lambda,a)\left[\mathbb{1} + \tfrac{1}{2} i\,\omega_{\rho\sigma} J^{\rho\sigma} - i\,\varepsilon_{\rho} P^{\rho}\right] U^{-1}(\Lambda,a)$$

$$= \mathbb{1} + \tfrac{1}{2} i \underbrace{(\Lambda\omega\Lambda^{-1})_{\mu\nu}}_{\substack{=\Lambda_{\mu}{}^{\rho}\omega_{\rho\sigma}(\Lambda^{-1})^{\sigma}{}_{\nu} \\ := \Lambda_{\mu}{}^{\rho}\Lambda_{\nu}{}^{\sigma}\omega_{\rho\sigma}}} J^{\mu\nu} - i \underbrace{(\Lambda\varepsilon - \Lambda\omega\Lambda^{-1}a)_{\mu} P^{\mu}}$$

$$
\begin{aligned}
&= (\Lambda_{\mu}{}^{\rho}\varepsilon_{\rho} - \Lambda_{\mu}{}^{\rho}\Lambda_{\nu}{}^{\sigma}\omega_{\rho\sigma} a^{\nu})\, P^{\mu} \\
&= \left[\Lambda_{\mu}{}^{\rho}\varepsilon_{\rho} - \tfrac{1}{2}(\omega_{\rho\sigma} - \omega_{\sigma\rho})\Lambda_{\mu}{}^{\rho}\Lambda_{\nu}{}^{\sigma} a^{\nu}\right] P^{\mu} \\
&= \left[\Lambda_{\mu}{}^{\rho}\varepsilon_{\rho} - \tfrac{1}{2}\Lambda_{\mu}{}^{\rho}\Lambda_{\nu}{}^{\sigma}\omega_{\rho\sigma} a^{\nu}\right] P^{\mu} + \tfrac{1}{2}\Lambda_{\nu}{}^{\sigma}\Lambda_{\mu}{}^{\rho}\omega_{\rho\sigma} a^{\mu} P^{\nu}
\end{aligned}
$$

Equating coefficients of $\omega_{\rho\sigma}$ and ε_{ρ}, we obtain:

$$
\begin{cases}
U(\Lambda,a) J^{\rho\sigma} U^{-1}(\Lambda,a) = \Lambda_{\mu}{}^{\rho}\Lambda_{\nu}{}^{\sigma}(J^{\mu\nu} - a^{\mu}P^{\nu} + a^{\nu}P^{\mu}) & (1) \\
U(\Lambda,a) P^{\rho} U^{-1}(\Lambda,a) = \Lambda_{\mu}{}^{\rho} P^{\mu} & (2)
\end{cases}
$$

(For homogeneous L.T. with $a=0$, it means that J is a tensor and P is a (four-)vector).

The rest of the proof does not require any new computational tricks, so we shorten it. Considering an infinitesimal L.T. with $\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu}$ and $a^{\mu} = \varepsilon^{\mu}$ in (1), we obtain after some algebra:

$$
\begin{aligned}
i\left[\tfrac{1}{2}\omega_{\mu\nu}J^{\mu\nu} - \varepsilon_{\mu}P^{\mu}, J^{\rho\sigma}\right] &= \omega_{\mu}{}^{\rho} J^{\mu\sigma} + \omega_{\nu}{}^{\sigma} J^{\rho\nu} - \varepsilon^{\rho}P^{\sigma} + \varepsilon^{\sigma}P^{\rho} \\
&= \left(\tfrac{1}{2}\eta^{\nu\rho}\omega_{\mu\nu}J^{\mu\sigma} - \tfrac{1}{2}\eta^{\mu\rho}\omega_{\mu\nu}J^{\nu\sigma}\right) + \left(\tfrac{-1}{2}\eta^{\sigma\mu}\omega_{\mu\nu}J^{\rho\nu} \right. \\
&\quad \left. - \tfrac{-1}{2}\eta^{\sigma\nu}\omega_{\mu\nu}J^{\rho\mu}\right) - \eta^{\mu\rho}\varepsilon_{\mu}P^{\sigma} + \eta^{\mu\sigma}\varepsilon_{\mu}P^{\rho}
\end{aligned}
$$

Matching the coefficients of $\omega_{\mu\nu}$ and ε_μ, we obtain:

$$\begin{cases} i[J^{\mu\nu}, J^{\rho\sigma}] = \eta^{\nu\rho} J^{\mu\sigma} - \eta^{\mu\rho} J^{\nu\sigma} - \eta^{\sigma\mu} J^{\rho\nu} + \eta^{\sigma\nu} J^{\rho\mu} \\ i[P^\mu, J^{\rho\sigma}] = \eta^{\mu\rho} P^\sigma - \eta^{\mu\sigma} P^\rho \end{cases}$$

Similarly, ② yields:

$$[P^\mu, P^\rho] = 0 .$$

The operators $J^{\mu\nu}$ and P^μ that satisfy these commutation relations form the Lie algebra of the Poincaré group, or the Poincaré algebra. Its structure constants are apparent.

II - Representations of the Homogeneous Lorentz Group

1) Relevance to Particle Physics (VERY BRIEF: Read [1] for the Physics!)

The special-relativistic physics of particles at the subatomic scale is well described by Quantum Field Theory (Q.F.T.). In QFT, the interaction $V(t)$ (classically the potential energy) defined as the difference between the full (the "total energy operator H") and the free-particle (the "kinetic energy operator H_0") Hamiltonian can be written as $V(t) = \int d^3x\, \mathcal{H}(\vec{x}, t)$, where the interaction density $\mathcal{H}(\vec{x}, t)$ is a scalar i.e.

$$U_0(\Lambda, a)\, \mathcal{H}(x)\, U_0^{-1}(\Lambda, a) = \mathcal{H}(\Lambda x + a) .$$

One way to do so is to construct $\mathcal{H}$ as a sum of products of fields Ψ whose components Ψ_ℓ are mixed by a position-independent matrix D under a L.T.:

$$U_0(\Lambda, a)\, \Psi_\ell\, U_0^{-1}(\Lambda, a) = \sum_{\bar\ell} D_{\ell\bar\ell}(\Lambda^{-1})\, \Psi_{\bar\ell}(\Lambda x + a) .$$

By applying a second L.T. $(\bar\Lambda, \bar a)$, we obtain

$$U_0(\bar{\Lambda},\bar{a})U_0(\Lambda,a)\psi_\ell U_0^{-1}(\Lambda,a)U_0^{-1}(\bar{\Lambda},\bar{a}) = \sum_{\bar{\ell}} D_{\ell\bar{\ell}}(\Lambda^{-1})\sum_{\bar{\bar{\ell}}} D_{\bar{\ell}\bar{\bar{\ell}}}(\bar{\Lambda}^{-1})\psi_{\bar{\bar{\ell}}}(\bar{\Lambda}(\Lambda x+a)+\bar{a})$$

$$\Leftrightarrow U_0(\bar{\Lambda}\Lambda,\bar{\Lambda}a+\bar{a})\psi_\ell U_0^{-1}(\bar{\Lambda}\Lambda,\bar{\Lambda}a+\bar{a}) = \sum_{\bar{\bar{\ell}}}\left(\sum_{\bar{\ell}} D_{\ell\bar{\ell}}(\Lambda^{-1})D_{\bar{\ell}\bar{\bar{\ell}}}(\bar{\Lambda}^{-1})\right)\psi_{\bar{\bar{\ell}}}(\bar{\Lambda}(\Lambda x+a)+\bar{a})$$

$$\Leftrightarrow \sum_{\bar{\bar{\ell}}} D_{\ell\bar{\bar{\ell}}}((\bar{\Lambda}\Lambda)^{-1})\psi_{\bar{\bar{\ell}}}(\bar{\Lambda}\Lambda x+\bar{\Lambda}a+\bar{a}) = \sum_{\bar{\bar{\ell}}}(D(\Lambda^{-1})D(\bar{\Lambda}^{-1}))_{\ell\bar{\bar{\ell}}}\psi_{\bar{\bar{\ell}}}(\bar{\Lambda}\Lambda x+\bar{\Lambda}a+\bar{a})$$

$$\Leftrightarrow D((\bar{\Lambda}\Lambda)^{-1}) = D(\Lambda^{-1})D(\bar{\Lambda}^{-1}),$$

so taking $\Lambda_1=\Lambda^{-1}$ and $\Lambda_2=\bar{\Lambda}^{-1}$, we find that the D-matrices furnish a <u>representation of the homogeneous Lorentz group</u>:

$$D(\Lambda_1)D(\Lambda_2) = D(\Lambda_1\Lambda_2).$$

When constructing a field ψ, the choice of the representation D under which it transforms is crucial for it dictates the properties of ψ i.e. the properties of the particle (and its antiparticle) that ψ describes.

2) General Representation of the Proper Orthochronous $SO^+(3,1)$ Subgroup

By the same reasoning as in Section I-3), the D-matrices near the identity are given by:

$$D(\Lambda) = \mathbb{1} + \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu},$$

where the generators satisfy:

$$[J_{\mu\nu},J_{\rho\sigma}] = i(J_{\rho\nu}\eta_{\sigma\mu} + J_{\mu\rho}\eta_{\nu\sigma} - J_{\sigma\nu}\eta_{\rho\mu} - J_{\mu\sigma}\eta_{\nu\rho}). \quad (3)$$

Note that since $D(\Lambda)$ acts on fields, it does not need to be unitary (unlike $U(\Lambda)$ which acts on state vectors). Therefore the $J_{\mu\nu}$'s are not Hermitian in general, and we may expect to find finite-dimensional representations of $SO^+(3,1)$ (despite its non-compactness).

We now define the vector operators $\vec{J}$ and $\vec{K}$ as:

$$\underset{\text{angular momentum}}{\vec{J}} = (J_1,J_2,J_3) := (J_{23},J_{31},J_{12}) \quad\text{and}\quad \underset{\text{boost}}{\vec{K}} = (K_1,K_2,K_3) := (J_{10},J_{20},J_{30})$$

Then, ③ implies: (See the brute force calculation in Reference [2])

$$[J_i, J_j] = i\,\varepsilon_{ijk} J_k\ , \quad [J_i, K_j] = i\,\varepsilon_{ijk} K_k$$

and $[K_i, K_j] = -i\,\varepsilon_{ijk} J_k$.

Had there not been a "−" sign in the last commutation relation, both the J_i's and the K_i would have satisfied the usual relations for the $so(3)$ algebra. The sign comes from $\eta_{00} = -1$ in the Minkowski metric η, which is the opposite of the 00-component of the 4-dimensional Euclidean space metric δ.

We now complexify $so(3,1)$ by setting:

$$\vec{A} := \frac{\vec{J} + i\vec{K}}{2} \quad \text{and} \quad \vec{B} := \frac{\vec{J} - i\vec{K}}{2}\ ,$$

in terms of which the commutation relations become:

$$[A_i, A_j] = i\,\varepsilon_{ijk} A_k\ , \quad [B_i, B_j] = i\,\varepsilon_{ijk} B_k$$

and $[A_i, B_j] = 0$.

Recalling that $so(3) \simeq su(2)$ with $su(2)_{\mathbb{C}} \simeq sl(2,\mathbb{C})$, we have found:

$$so(3,1)_{\mathbb{C}} \simeq sl(2,\mathbb{C}) \oplus sl(2,\mathbb{C})\ .$$

(Remarks: - In fact, $so(3,1) \simeq sl(2,\mathbb{C})$ so it is not surprising that, complexifying $so(3,1)$, we obtain two copies of $sl(2,\mathbb{C})$;
- In Euclidean space, $so(4) \simeq so(3) \oplus so(3)$ by the same reasoning (see [4])
- $\vec{J}$, $\vec{A}$ and $\vec{B}$ are Hermitian but $\vec{K}$ is anti-Hermitian.)

The rest of the derivation is explained in any Quantum Mechanics textbook. We take $(\vec{A})_{a'b',ab} = \delta_{b'b}\, \vec{J}^{(A)}_{a'a}$ and $(\vec{B})_{a'b',ab} = \delta_{a'a}\, \vec{J}^{(B)}_{b'b}$, where a and b are integers and/or half-integers indices running over the values $a = -A, -A+1, \dots, +A$ ("spin projection") and $b = -B, -B+1, \dots, B$, and $\vec{J}^{(A)}$ and $\vec{J}^{(B)}$ are the usual angular momentum (but here it is spin j!) matrices for $j_1 = A$ and $j_2 = B$, respectively. Therefore, a representation of $SO^{+}(3,1)$ can be denoted by (A,B) and has dimension $(2A+1)(2B+1)$.

Finally, since $\vec{J} = \vec{A} + \vec{B}$, a field ψ that transforms under the (A,B) representation has components ψ_{ab} that rotate like objects of total spin j where:

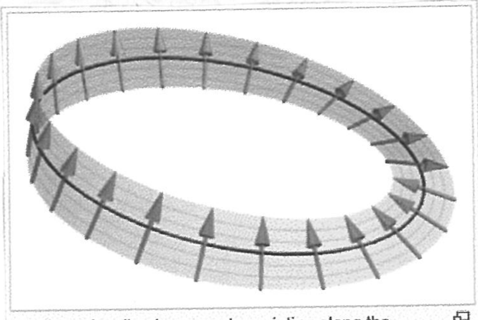
$$j = A+B,\ A+B-1,\ \dots,\ |A-B|$$

by the usual rule of addition of angular momenta.
↳ (Similar to the calculation of the norm of the addition of 2 (anti-)parallel vectors.)

3) Examples

- A $(0,0)$ field can only have $j=0$, hence it is scalar. The scalar representation has $D(\Lambda) = 1$.
- Both the $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ fields have $j = \frac{1}{2}$ and may describe the same particle (and its anti-particle), but with opposite chiralities. (Chirality is an abstract concept that will be defined next, but a simple case to keep in mind is the one of a massless particle for which it is the same as helicity.) These fields correspond to the left-handed and the right-handed Weyl spinor representations, respectively.

Interesting visualization of a spinor:



A spinor visualized as a vector pointing along the Möbius band, exhibiting a sign inversion when the circle (the "physical system") is rotated through a full turn of 360°.[nb 1]

Source: Wikipedia.

The $2\frac{1}{2}+1 = 2$-dimensional left-handed and right handed spinors ψ_L and ψ_R are the top and bottom component of the Dirac spinor ψ i.e. ψ is a 4-dimensional bispinor with $\psi = (\psi_L, \psi_R)$ and it transforms under the $(\frac{1}{2},0) \oplus (0,\frac{1}{2})$ representation.

We now briefly explain how ψ_L and ψ_R arise. We first define the Dirac matrices

γ^μ that satisfy the following anticommutation relations:

$$\{\gamma^\mu, \gamma^\nu\} := \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} \qquad (4)$$

and we set:

$$\tilde{\mathcal{J}}^{\mu\nu} = -\frac{i}{4}[\gamma^\mu, \gamma^\nu] \qquad (5)$$

It is straightforward (see [3]) to show that the $\tilde{\mathcal{J}}_{\mu\nu}$'s satisfy the commutation relations (3) of $so(3,1)$. Therefore we can write $\tilde{\mathcal{J}}^{\mu\nu} = \mathcal{J}^{\mu\nu}$ for this particular representation, and we deduce that products of γ-matrices can act on fields belonging to that representation.

In 3+1-dimensional spacetime, we can show that (see [3], 5.4) we can take the irreducible γ-matrices to be 4×4, and choose:

$$\gamma^0 = -i\begin{pmatrix} 0_2 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0_2 \end{pmatrix} \quad \text{and} \quad \vec{\gamma} = -i\begin{pmatrix} 0_2 & \vec{\sigma} \\ -\vec{\sigma} & 0_2 \end{pmatrix},$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the Pauli matrices. (Of course, (4) is obeyed.) This allows us to calculate the explicit forms of the $\mathcal{J}^{\mu\nu}$ generators, which are block-diagonal:

$$\mathcal{J}^{ij} = \frac{1}{2}\varepsilon_{ijk}\begin{pmatrix} \sigma_k & 0_2 \\ 0_2 & \sigma_k \end{pmatrix} \quad \text{and} \quad \mathcal{J}^{i0} = +\frac{i}{2}\begin{pmatrix} \sigma_i & 0_2 \\ 0_2 & -\sigma_i \end{pmatrix}$$

Finally, we define the fifth γ-matrix γ_5 as:

$$\gamma_5 := -i\gamma^0\gamma^1\gamma^2\gamma^3 = -i(-i)^4 \begin{pmatrix} 0&0&1&0 \\ 0&0&0&1 \\ 1&0&0&0 \\ 0&1&0&0 \end{pmatrix} \begin{pmatrix} 0&0&0&1 \\ 0&0&1&0 \\ 0&-1&0&0 \\ -1&0&0&0 \end{pmatrix} \begin{pmatrix} 0&0&0&-i \\ 0&0&i&0 \\ 0&i&0&0 \\ -i&0&0&0 \end{pmatrix} \begin{pmatrix} 0&0&1&0 \\ 0&0&0&-1 \\ -1&0&0&0 \\ 0&+1&0&0 \end{pmatrix}$$

$$= -i \begin{pmatrix} 0&-1&0&0 \\ -1&0&0&0 \\ 0&0&0&1 \\ 0&0&1&0 \end{pmatrix} \begin{pmatrix} 0&-i&0&0 \\ -i&0&0&0 \\ 0&0&0&-i \\ 0&0&-i&0 \end{pmatrix}$$

$$= \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&-1&0 \end{pmatrix} = \begin{pmatrix} \mathbb{1}_2 & 0_2 \\ 0_2 & -\mathbb{1} \end{pmatrix}$$

We immediately see that $[\gamma^{\mu\nu}, \gamma_5] = 0$, which means that γ_5 is invariant under rotation and boosts. Setting $P_\pm = \frac{\mathbb{1}_4 \pm \gamma_5}{2}$, we see that:

$$P_+ = \begin{pmatrix} \mathbb{1}_2 & 0_2 \\ 0_2 & 0_2 \end{pmatrix} \quad \text{and} \quad P_- = \begin{pmatrix} 0_2 & 0_2 \\ 0_2 & \mathbb{1}_2 \end{pmatrix}$$

with $P_\pm^2 = P_\pm$ (and $P_\pm P_\mp = 0$). Therefore, $P_\pm$ are projection operators and

$$\psi_L = P_+ \psi \quad \text{and} \quad \psi_R = P_- \psi.$$

So we have found a Lorentz-invariant definition of chirality. It has many important consequences in theoretical physics.

- The $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ representation are the "simplest non-trivial" representation. The next one to consider is $\underline{(\frac{1}{2},\frac{1}{2})}$, which actually is $(\frac{1}{2},\frac{1}{2}) = (\frac{1}{2},0) \otimes (0,\frac{1}{2})$. It has dimension 4, which corresponds to the field components with $j_1 = 1$ $(m_1 = -1, 0, 1)$ and $j_2 = 0$. It is called the <u>4-vector representation</u>, because the $j_1 = 1$ components correspond to the (rotating) spatial part $\vec{v}$ of a 4-vector v, while $j_2 = 0$ corresponds to the time-component (i.e. the 0-component) of v. This representation has $\underline{D(\Lambda)^\mu{}_\nu = \Lambda^\mu{}_\nu}$.

References :

[1] *The Quantum Theory of Fields, Volume 1: Foundations* by S. Weinberg

[2] QFT-I Sheet by Prof. Isidori at ETH and UZH:
https://www.uzh.ch/cmsssl/physik/dam/jcr:6cfed0b7-2440-4535-9fbc-4828c945ade8/Solution03.pdf

[3] QFT lecture notes by D. Tong at Cambridge: http://www.damtp.cam.ac.uk/user/tong/qft/four.pdf

[4] About **so**(4) by Prof. Haber at UCSC:
http://scipp.ucsc.edu/~haber/archives/physics251_13/groups13_sol4.pdf

[5] Wikipedia, Wikiversity and Physics (or Math) Stack Exchange are always useful!