Def.

A Lie algebra $\mathcal{L}$ is said to be Abelian if $[X, Y] = 0$ for any $X, Y \in \mathcal{L}$.

e.g. ∘ trivial Lie algebra (It has only one element 0)

Now I denote $[\mathcal{L}, \mathcal{L}] = \langle \{[h, k] \mid h, k \in \mathcal{L}\} \rangle$ (a Lie subalgebra generated by Lie brackets)

Def. ∘ Let $\mathcal{L}$ be a Lie algebra over $\mathbb{K}$ ($\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$).

The lower central series of $\mathcal{L}$ is the descending chain of subspaces

$$\mathcal{L} \supseteq \mathcal{L}^1 := [\mathcal{L}, \mathcal{L}] \supseteq \mathcal{L}^2 := [\mathcal{L}, \mathcal{L}^1] \supseteq \cdots \supseteq \mathcal{L}^n := [\mathcal{L}, \mathcal{L}^{n-1}] \supseteq \cdots$$

∘ The derived series of $\mathcal{L}$ is the descending chain of subspaces

$$\mathcal{L} \supseteq \mathcal{L}^{(1)} := [\mathcal{L}, \mathcal{L}] \supseteq \mathcal{L}^{(2)} := [\mathcal{L}^{(1)}, \mathcal{L}^{(1)}] \supseteq \cdots \supseteq \mathcal{L}^{(n)} := [\mathcal{L}^{(n-1)}, \mathcal{L}^{(n-1)}] \supseteq \cdots$$

Def. $\mathcal{L}$ is called nilpotent if $\mathcal{L}^n = 0$ for some $n \geq 1$.

$\mathcal{L}$ is called solvable if $\mathcal{L}^{(n)} = 0$ for some $n \geq 1$

In fact $\{\text{Abelian}\} \subsetneq \{\text{nilpotent}\} \subsetneq \{\text{solvable}\}$ as mentioned in the lecture. (To prove $\{\text{nilpotent}\} \subset \{\text{solvable}\}$, it is enough to show that $\mathcal{L}^{(n)} \leq \mathcal{L}^n$ for $n \geq 1$ by induction.)

e.g. Let $t_n := \{\text{upper triangular matrices}\}$ $\rightsquigarrow$ $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ & & \ddots & \vdots \\ 0 & & & a_{nn} \end{bmatrix}$

$s_n := \{\text{strictly upper triangular matrices}\}$ $\rightsquigarrow$ $\begin{bmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ & 0 & a_{23} & \cdots & a_{2n} \\ & & \ddots & \ddots & \vdots \\ 0 & & & & 0 \end{bmatrix}$

($s_n, t_n \leq \mathfrak{gl}_n(\mathbb{K})$: general linear Lie algebra)

Then t_n is a solvable Lie algebra but not nilpotent.

s_n is a nilpotent Lie algebra.

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Thm. (Engel's Theorem)

Let $\mathcal{L}$ be a finite-dimensional Lie algebra.
Then $\mathcal{L}$ is nilpotent if and only if for each $x \in \mathcal{L}$ $(ad_x)^n = 0$ for some $n \geq 1$.

Thm. (Cartan's Criterion)

Let $\mathcal{L}$ be a subalgebra of $gl(V)$ where V is finite dimensional.
Then $\mathcal{L}$ is solvable if and only if $Tr(xy) = 0$ for all $x \in [\mathcal{L}, \mathcal{L}]$ and $y \in \mathcal{L}$.

Cor. A Lie algebra $\mathcal{L}$ is solvable if and only if $Tr(ad_x\, ad_y) = 0$ for all $x \in [\mathcal{L}, \mathcal{L}]$, $y \in \mathcal{L}$
i.e. the Killing form on $\mathcal{L}$ is identically zero.

Rmk.
- a nilpotent Lie algebra is an analog of a nilpotent group.
  (A nilpotent group G has a lower central series terminating in the trivial subgroup after finite steps i.e.
  $G =: G_0 \rhd G_1 \rhd \cdots \rhd G_n = \{e\}$ where $G_{k+1} = [G_k, G]$)
- a solvable Lie algebra is an analog of a solvable group.
  (A solvable group G has a descending normal series reaching the trivial subgroup.
  $G =: G^{(0)} \rhd G^{(1)} \rhd \cdots \rhd G^{(n)} = \{e\}$ where $G^{(k+1)} = [G^{(k)}, G^{(k)}]$)

($[G, G]$: the commutator subgroup of a group G)