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Exercise

Let G be a group and let $Aut(G) = \{\text{automorphisms of } G\}$.
For $b \in G$, let $ad_b \in Aut(G)$, $ad_b(c) = bcb^{-1}$ $(\forall c \in G)$
Show that:

1) $Aut(G)$ is a group.
2) $\{ad_b \mid b \in G\} \triangleleft Aut(G)$
3) $\phi : G \longrightarrow \{ad_b \mid b \in G\}$, $b \longmapsto ad_b$ is a homomorphism.

1) Let $x, y \in G$ and $f, g \in Aut(G)$. If $x \neq y$, then $g(x) \neq g(y)$
$f(g(x)) \neq f(g(y))$ i.e. $(f \circ g)(x) \neq (f \circ g)(y)$ because f, g are injective.

Let $a \in G$. f is surjective so there exists $\alpha \in G$ s.t. $f(\alpha) = a$. Furthermore, g is also surjective so there exists $\beta \in G$ s.t. $g(\beta) = \alpha$. Thus $a = f(\alpha) = f(g(\beta)) = (f \circ g)(\beta)$.
Therefore $f \circ g : G \rightarrow G$ is bijective.
Indeed for all $x, y \in G$

$$(f \circ g)(xy) = f(g(xy)) = f(g(x)\, g(y)) = f(g(x))\, f(g(y)) = (f \circ g)(x)\, (f \circ g)(y)$$

$\therefore f \circ g \in Aut(G)$.

Let $(f_1, g_1), (f_2, g_2) \in Aut(G) \times Aut(G)$ with $(f_1, g_1) = (f_2, g_2)$ $\Leftrightarrow$ $f_1 = f_2$ and $g_1 = g_2$
Then For all $x \in G$, $(f_1 \circ g_1)(x) = (f_2 \circ g_2)(x)$
Hence, $\rho : Aut(G) \times Aut(G) \longrightarrow Aut(G)$ is well-defined.

$$\begin{array}{ccc} \rho : Aut(G) \times Aut(G) & \longrightarrow & Aut(G) \\ \Psi & & \Psi \\ (f, g) & \longmapsto & f \circ g \end{array}$$

Now I will show that $Aut(G)$ is a group with the operation $\circ$.

(i) Associativity

Let $f, g, h \in Aut(G)$. For all $x \in G$

$$(f\circ(g\circ h))(x) = f((g\circ h)(x)) = f(g(h(x))) = (f\circ g)(h(x))$$
$$= ((f\circ g)\circ h)(x) \qquad \therefore\ f\circ(g\circ h) = (f\circ g)\circ h$$

(ii) Identity

The identity element of $Aut(G)$ is the identity map

$$id: G \to G \qquad \text{Indeed, for all } x\in G,\ f\in Aut(G)$$
$$x \mapsto x$$

$$(f\circ id)(x) = f(id(x)) = f(x) = id(f(x)) = (id\circ f)(x) \qquad \therefore\ f\circ id = id\circ f = f$$

(iii) Inverse

The inverse element of $f\in Aut(G)$ is $f^{-1}\in Aut(G)$

(There is the inverse map f^{-1} of f because f is bijective and f^{-1} is also bijective)

Indeed, for all $x\in G$ $\quad (f\circ f^{-1})(x) = f(f^{-1}(x)) = x = id(x) \quad \therefore\ f\circ f^{-1} = id$

Therefore $Aut(G)$ is a group. $\square$

2) (i) $\{ad_b \mid b\in G\}$ is a subgroup of $Aut(G)$

$$(ad_a\circ ad_b)(x) = ad_a(ad_b(x)) = ad_a(bxb^{-1}) = a(bxb^{-1})a^{-1}$$
$$= (ab)x(ab)^{-1} = ad_{ab}(x) \in \{ad_c \mid c\in G\} \qquad (x\in G)$$

Indeed, $\quad ad_a\circ ad_b \in \{ad_c \mid c\in G\}$

- The identity element is ad_e (e: the identity of G)

Indeed, for all $x\in G$, $ad_b\in\{ad_b \mid b\in G\}$

$$(ad_b\circ ad_e)(x) = ad_b(ad_e(x)) = ad_b(exe^{-1}) = ad_b(x) = e(ad_b(x))e^{-1}$$

($\times$ $e^{-1}=e$, $exe^{-1}=x$) $\qquad = (ad_e\circ ad_b)(x)$

$\therefore\ ad_b \circ ad_e = ad_e \circ ad_b = ad_b$

· The inverse element of ad_b is $ad_{b^{-1}}$

Indeed for all $x \in G$. $\quad ((b^{-1})^{-1} = b)$

$$(ad_b \circ ad_{b^{-1}})(x) = ad_b(b^{-1}xb) = b(b^{-1}xb)b^{-1} = x = ad_e(x)$$

(ii) $\{ad_b \mid b \in G\}$ is a normal subgroup of $Aut(G)$

Let $\sigma \in \{ad_b \mid b \in G\}$, $\tau \in Aut(G)$. Then there exists $g \in G$ s.t. $\sigma = ad_g$. For all $x \in G$

$$(\tau \circ ad_g \circ \tau^{-1})(x) = \tau(ad_g(\tau^{-1}(x))) = \tau(g\tau^{-1}(x)g^{-1})$$

$$= \tau(g)\,x.\,\tau(g^{-1}) = \tau(g)\,x\,(\tau(g))^{-1} = ad_{\tau(g)}(x)$$

$$\tau \circ ad_g \circ \tau^{-1} = ad_{\tau(g)} \in \{ad_b \mid b \in G\}$$

$\therefore\ \{ad_b \mid b \in G\} \triangleleft Aut(G)$ $\square$

3) Let $x, y \in G$. For all $z \in G$.

$$ad_{xy}(z) = (xy)z(xy)^{-1} = (xy)z(y^{-1}x^{-1}) = x(yzy^{-1})x^{-1}$$

$$= x(ad_y(z))x^{-1} = ad_x(ad_y(z)) = (ad_x \circ ad_y)(z)$$

$$\therefore\ ad_{xy} = ad_x \circ ad_y$$

Then. $\phi(xy) = ad_{xy} = ad_x \circ ad_y = \phi(x) \circ \phi(y)$. Indeed. ϕ is a homomorphism. $\square$