Hilbert spaces

Def A Hilbert space $\mathcal{H}$ is a complex vector space with

- A scalar product satisfying for $f, g \in \mathcal{H}$:

  $\langle f, g \rangle = \overline{\langle g, f \rangle}$ ← complex conjugate

  $\langle f, \alpha g + h \rangle = \alpha \langle f, g \rangle + \langle f, h \rangle \quad \forall \alpha \in \mathbb{C}$

  $\langle f, f \rangle \geq 0$ and $\langle f, f \rangle = 0$ iff $f = 0$

  0-element of the vector space

  This scalar product defines a norm

  $\| f \| := \sqrt{\langle f, f \rangle}$.

- The vector space is complete for this norm:

  If $\{f_n\}_{n \in \mathbb{N}} \subset \mathcal{H}$ is a sequence of elements of $\mathcal{H}$ such that $\forall \varepsilon > 0$ there exists $N \in \mathbb{N}$ with $\| f_n - f_m \| < \varepsilon$ for $n, m > N$, ⊛

  then $\exists f_\infty \in \mathcal{H}$ such that $\| f_n - f_\infty \| \xrightarrow{n \to \infty} 0$.

A sequence satisfying ⊛ is called a Cauchy sequence, and the property above means that any Cauchy sequence has a limit in $\mathcal{H}$.

⚠ The scalar product is linear in the second argument, other conventions are possible.

Examples

1) $\mathcal{H} = \mathbb{C}^n = \{ (a_1, \dots, a_n) \text{ with } a_j \in \mathbb{C} \ \forall j = 1, \dots, n \}$

The scalar product for $\mathbf{a}, \mathbf{b} \in \mathbb{C}^n$ is

$\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{j=1}^{n} \overline{a_j}\, b_j$. Then, one

has $\| \mathbf{a} \| = \sqrt{|a_1|^2 + |a_2|^2 + \dots + |a_n|^2}$

it is the modulus of the complex number a_2

2) $\mathcal{H} = \ell^2(\mathbb{Z}) = \{ \mathbf{a} = (a_j)_{j \in \mathbb{Z}} \mid \sum_{j \in \mathbb{Z}} |a_j|^2 < \infty \}$

ℓ^2 sequence on $\mathbb{Z}$

The scalar product for $\mathbf{a}, \mathbf{b} \in \ell^2(\mathbb{Z})$

is $\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{j \in \mathbb{Z}} \overline{a_j}\, b_j$.

integral of Lebesgue

3) $\mathcal{H} = L^2(\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{C} \mid \int_{\mathbb{R}} |f(x)|^2 \, dx < \infty \}$

The scalar product for $f, g \in \mathcal{H}$ is

$\langle f, g \rangle = \int \overline{f(x)}\, g(x)\, dx$

complex conjugate of $f(x)$.

called the set of L^2-functions on $\mathbb{R}$.

This Hilbert space is very important for quantum mecanics.

Important inequalities (for any Hilbert space)

$$|\langle f, g \rangle| \leq \|f\| \, \|g\|$$

Cauchy - Schwarz inequality

$$\|f + g\| \leq \|f\| + \|g\|$$

$$\|f + g\|^2 \leq 2\|f\|^2 + 2\|g\|^2$$

$$\big| \|f\| - \|g\| \big| \leq \|f - g\|$$

Usually, we consider Hilbert spaces which are separable: There exists a countable orthonormal basis for $\mathcal{H}$:

$\exists \, \{e_j\}_{j \in \mathbb{N}}$ with $\|e_j\| = 1$ and $\langle e_j, e_k \rangle = 0$ if $j \neq k$ and such that any $f \in \mathcal{H}$ can be written as $f = \sum_{j \in \mathbb{N}} c_j e_j$ with $c_j \in \mathbb{C}$.

The dimension of $\mathbb{C}^n$ is n, but $\ell^2(\mathbb{Z})$ or $L^2(\mathbb{R})$ are of dimension ∞ (the basis contains an infinite number of elements).