
Homework 4

Let us recall the polar coordinates $(r, \theta) \in [0, \infty) \times [0, 2\pi)$ which allow one to give an alternative description of all the points of \mathbb{R}^2 . The relations between the usual coordinates $(x, y) \in \mathbb{R} \times \mathbb{R}$ are

$$x = r \cos(\theta) \quad \text{and} \quad y = r \sin(\theta).$$

Exercise 1 Consider the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by

$$f(x, y) := \begin{cases} \frac{x^3}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

1. By using the polar coordinates, study the continuity of f at $(0, 0)$,
2. Show that f is differentiable with continuous partial derivatives on $\mathbb{R}^2 \setminus \{(0, 0)\}$,
3. Show that f admits in $(0, 0)$ derivatives in all directions,
4. Show that f is not differentiable at $(0, 0)$.

Exercise 2 Consider the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by

$$f(x, y) := \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

and compute its second derivatives $\partial_j \partial_k f$ for any $j, k \in \{x, y\}$. Are these functions continuous ?

Exercise 3 (Geometrical interpretation of the gradient) a) Consider the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by $f(x, y) = x^2 + 4y^2$.

- (i) Compute the gradient of f at any point (x, y) ,
- (ii) For $k > 0$, describe the k -level set L_k , and for this level set express y as a function of x and k ,
- (iii) For any $(x, y) \in \mathbb{R}^2$ such that $f(x, y) = k$, show that the gradient of f at (x, y) is orthogonal to the curve described by L_k .

b) More generally, let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a function of class C^1 and let $X \in \mathbb{R}^n$ such that $[\nabla f](X) \neq \mathbf{0}$. Let $k \in \mathbb{R}$ be given by $k := f(X)$ and consider the k -level set L_k . This k -level set can be considered (at least locally) as a surface of dimension $n - 1$ in \mathbb{R}^n . Show that $[\nabla f](X)$ is perpendicular to the surface L_k . For that purpose, we can consider any parametric curve $\varphi : (-1, 1) \rightarrow L_k$ with $\varphi(0) = X$ and show that $[\nabla f](X)$ is perpendicular to it at the point X .

Exercise 4 (The chain rule) Let Ω be an open set in \mathbb{R}^n and let $f : \Omega \rightarrow \mathbb{R}$ be differentiable. Let (a, b) be an open interval in \mathbb{R} , and consider $\varphi : (a, b) \rightarrow \Omega$ be a parametric curve which is differentiable. Show that the following equality holds:

$$f(\varphi(t))' := \frac{df(\varphi(t))}{dt} = [\nabla f](\varphi(t)) \cdot \varphi'(t).$$