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**Homework 3**

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**Exercise 1** Let  $\Omega \subset \mathbb{R}^2$  and consider the functions  $f_i : \Omega \rightarrow \mathbb{R}$  defined for  $(x, y) \in \Omega$  by

$$a) f_1(x, y) = xy, \quad b) f_2(x, y) = (x+1)(y+3) \quad c) f_3(x, y) = \frac{xy}{x^2+y^2} \quad d) f_4(x, y) = \frac{x+y}{x-y}.$$

1. Determine the maximal domain  $\Omega$  on which these functions are well defined,
2. Sketch the  $k$ -level sets for these functions.

**Exercise 2** Consider the following functions defined on  $\mathbb{R}^2 \setminus \{(0, 0)\}$

$$a) f_1(x, y) = \frac{x^2 - y^2}{x^2 + y^2}, \quad b) f_2(x, y) = \frac{xy}{x^2 + y^2}, \quad c) f_3(x, y) = \frac{1}{x^2 + y^2 + 1}.$$

For each of them compute the limits  $\lim_{x \rightarrow 0} (\lim_{y \rightarrow 0} f_i(x, y))$ ,  $\lim_{y \rightarrow 0} (\lim_{x \rightarrow 0} f_i(x, y))$ , and  $\lim_{(x,y) \rightarrow (0,0)} f_i(x, y)$ . Discuss your result.

**Exercise 3** Consider the function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  defined by

$$f(x, y) := \begin{cases} \frac{x^2 y}{x^4 - 2x^2 y + 3y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

1. Study the limit  $(x, y) \rightarrow (0, 0)$  along the path of equation  $y = mx$  for any  $m \in \mathbb{R}$ ,
2. Study the limit  $(x, y) \rightarrow (0, 0)$  along the path of equation  $y = x^2$ ,
3. Show that  $f$  is not continuous at  $(0, 0)$ .

**Exercise 4** Compute the partial derivatives of the functions of Exercise 1 on their respective domain. Compute also the partial derivatives of the following functions:

(i)  $f : \mathbb{R}_+ \times \mathbb{R} \ni (x, y) \mapsto f(x, y) = x^y \in \mathbb{R}$ ,

(ii)  $g : (\mathbb{R}_+)^3 \rightarrow \mathbb{R}$  given by  $g(x, y, z) = x^3 y^2 + \sin(xz) - \ln(xyz)$ .