
Homework 2

Exercise 1 Let $f : (a, b) \rightarrow \mathbb{R}^d$ be a parametric curve of class C^1 , and let $\varphi : (c, d) \rightarrow (a, b)$ be of class C^1 and strictly increasing, with $\varphi(c) = a$ and $\varphi(d) = b$. Show that the equality $L_f = L_{f \circ \varphi}$ holds, or in other words show that the length of the curve does not depend on the parametrization.

Exercise 2 Write a parametric equation for the tangent line at any point of the curve given by

$$g : \mathbb{R} \ni t \mapsto (e^{3t}, e^{-3t}, t, 1) \in \mathbb{R}^4.$$

Exercise 3 Consider the spiral in \mathbb{R}^3 defined by the function

$$f : \mathbb{R} \ni t \mapsto (\cos(t), \sin(t), t) \in \mathbb{R}^3.$$

Determine the equation of the plane perpendicular to the spiral for any $t \in \mathbb{R}$.

Exercise 4 Find the length of the spiral of the previous exercise between $t = 0$ and $t = 1$.

Exercise 5 Consider the parametric curve given by

$$c : \mathbb{R} \ni t \mapsto (e^t \cos(t), e^t \sin(t))$$

Show that the tangent vector to the curve makes a constant angle with the position vector, *i.e.* with the vector $c(\cdot)$.

Exercise 6 Consider a parametric curve $f : \mathbb{R} \rightarrow \mathbb{R}^d$ of class C^2 , and us call the osculating plane at t the plane passing by $f(t)$ and defined by the two vectors $f'(t)$ and $f''(t)$. Obviously this plane is well defined only if these two vectors are not parallel.

(i) Determine the osculating plane at any t for the spiral defined in Exercise 3.

(i) More generally, for any parametric curve f of class C^2 and for any diffeomorphism φ of class C^2 , show that the osculating plane defined by f or by the new parametric curve $f \circ \varphi$ **at the same point** are equal.